Exercise 10.1

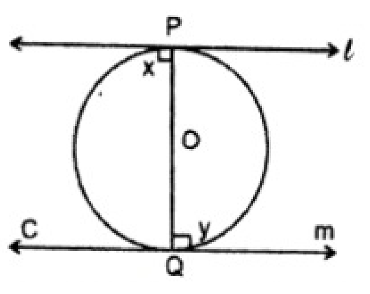
1. Prove that the tangents drawn at the ends of a diameter in a given circle must be parallel and conversely.

Solution:

Given:

Let ℓ and m be two tangents to the circle at the end points of a diameter $\overline{PQ}$

.



To prove: $\ell \parallel m$

Proof:

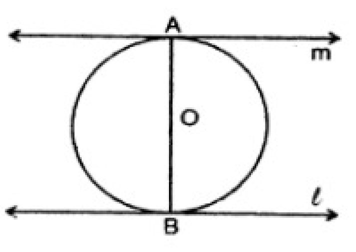
Statements	Reasons
$OP \perp \ell$, $OQ \perp m$ $\angle x = 90^\circ$, $\angle y = 90^\circ$ $\Rightarrow m\angle x = m\angle y = 90^\circ$ But: $m\angle x$ and $m\angle y$ are alternate angles. Hence, $\ell \parallel m$	$\therefore$ A tangent at any point of a circle is $\perp$ to the radius through the point of contact.

Conversely: parallel tangents of a circle must pass through its center.

Given:

Let ℓ and m are tangent to the circle at the ends of diameter $\overline{AB}$.

To the center O, and $AB \perp \ell$ and $AB \perp m$.



To prove:

$\overline{AB}$ passes through the center (diameter)

Proof:

If $\overline{AB}$ does not pass through the center join $\overline{OB}$.

$\overline{OB}$ is radius and ℓ is a tangent at B.

So that

$\overline{OB} \perp \ell$ or

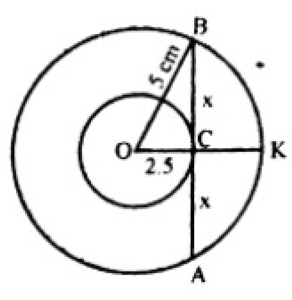
But $\overline{AB} \perp \ell$. (given)

$\therefore \overline{OB}$ coincides with $\overline{AB}$.

Hence, $\overline{AB}$ passes through the center.

2. The diameters of two concentric circles are 10 cm and 5cm respectively. Look for the length of any chord of the outer circle which touches the inner one.

Solution:



In a triangle OCB.

$$\left(\overline{OB}\right)^2 = \left(\overline{OC}\right)^2 + \left(\overline{CB}\right)^2$$

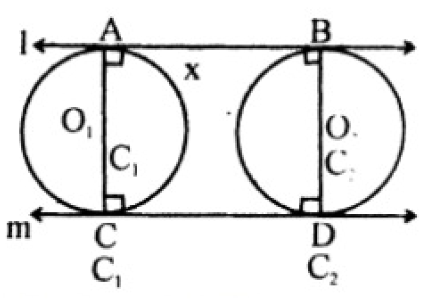
$$\begin{aligned} \Rightarrow \left(\overline{CB}\right)^2 &= \left(\overline{OB}\right)^2 - \left(\overline{OC}\right)^2 \\ &= (5)^2 - (2.5)^2 \\ &= 25 - 6.25 \\ &= 18.75 \end{aligned}$$

$$\Rightarrow \quad \overline{CB} = \sqrt{18.75}$$

$$\overline{AB} = 2\overline{CB} = 2\sqrt{18.75} = 8.7\text{cm}$$

3. AB and CD are the common tangents drawn to the pair of circles. If A and C are the points of tangency of 1st circle where B and D are the points of tangency of 2nd circle, then prove that AC II BD.

Solution:



Given:

Two circle C_1 and C_2. Points of tangency of C_1

is A and C and points of tangency of C_2 is B and D

To Prove:

$AC \parallel BC$

Proof:

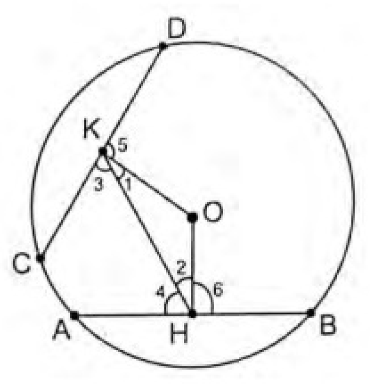
Statements	Reasons
In circle "C_1"	
And $\ell \parallel m$	Tangent is perpendicular to the circle
$m\angle CAB = 90^\circ$ _______(i)	
and in circle "C_2"	
$\ell \parallel m$	
and $m\angle ABC = 90^\circ$ _______(ii)	Proved
$\Rightarrow$ $\angle CAB \cong \angle ABD$	Tangent is perpendicular to the circle
Similarly, $\angle ACD \cong BDC$	by (i) 4 (ii)

Therefore: ABCD is rectangle $\therefore \overline{AC} \parallel \overline{BC}$	Parallel sides of a rectangle.

EXERCISE 10.2

I. AB and CD are two equal chords in a circle with center O. H and K are respectively the mid points of the chords. Prove that HK makes equal angles with AB and CD.

Solution:



Given:

$\overline{AB}$ and $\overline{CD}$ are equal chords of a circle with center O.

To prove:

(i) $m\angle AHK = m\angle CKH$

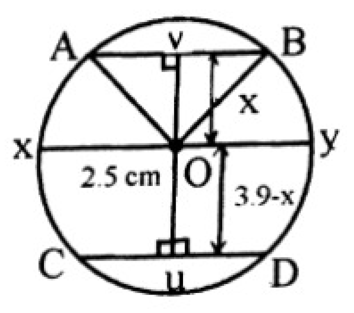
(ii) $m\angle BHK = m\angle DKH$

Statements	Reasons
In $\triangle HOK$	
$m\overline{OH} = m\overline{OK}$	Radii of the circle.
$\therefore m\angle 1 = m\angle 2$ ____(i)	
Also $m\angle 5 = m\angle 6$ ____(ii)	Each of 90°
$m\angle 1 + m\angle 5 = m\angle 2 + m\angle 6$	adding (i) and (ii)
$m\angle DKH = m\angle BHK$	Proved
$m\angle AHO = m\angle CKO$ ____(iii)	Each of 90°
$m\angle 2 = m\angle 1$ ____(iv)	
$m\angle AHO - m\angle 2 = m\angle CKO - m\angle 1$	Subtract (iv) from (iii)
$m\angle AHK = m\angle CKH$	Proved

2. The radius of a circle is 2.5 cm. AB and CD are two chords 3.9cm apart.

If mAB = 1.4 cm, then measure' the other chord.

Solution:



Given:

$m\overline{OY} = m\overline{OX} = 2.5\text{cm}$

$m\overline{UV} = 3.9\text{cm}$

$m\overline{AB} = 1.4\text{cm}$

Required:

$m\overline{CD} = ?$

In triangle OAV

$m\overline{OA}^2 = m\overline{OV}^2 + m\overline{VA}^2$

$2.5^2 = x^2 + (0.7)^2$

$\Rightarrow x^2 = 2.5^2 - 0.7^2$

$= 625 - 0.49 = 5.76$

$x = 2.4$ cm

$m\overline{OU} = 3.9 - 2.4 = 1.5$ cm

In $\triangle OUC$

$m\overline{OC}^2 = m\overline{OU}^2 + m\overline{CU}^2$

$2.5^2 = 1.5^2 + m\overline{CU}^2$

$\Rightarrow m\overline{CU}^2 = 2.5^2 - 1.5^2$

$= 6.25 - 2.25 = 4$

$\overline{CU}^2 = 4 \Rightarrow \overline{CU} = \sqrt{4} = 2$

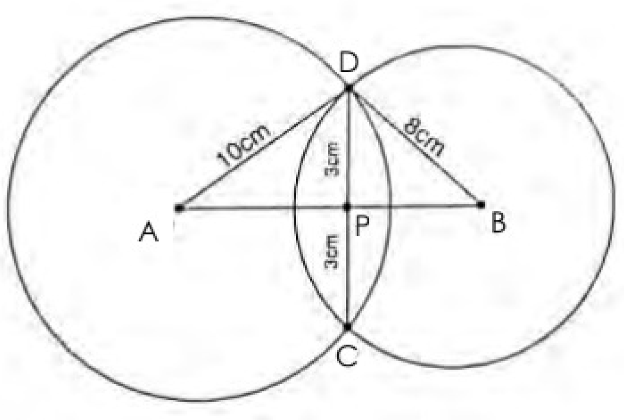
$m\overline{CD} = m\overline{CU} + m\overline{UD}$

$m\overline{CD} = 2 + 2$

$\overline{CD} = 4\text{cm}$

3. The radii of two intersecting circles are 10cm and 8cm. If the length of their common chord is 6cm then find the distance between the centers.

Solution:



Given:

$m\overline{AD} = 10\text{cm}$, $m\overline{BD} = 8\text{cm}$,

$m\overline{DC} = 6\text{cm}$

Required:

$m\overline{AP} = ?$, $m\overline{PB} = ?$

In $\triangle$ ADP.

$m\overline{AD}^2 = m\overline{DP}^2 + m\overline{AP}^2$ $\quad \because m\overline{AD} = m\overline{AP}$

$(10)^2 = (3)^2 + mAP^2$

$\Rightarrow mAP^2 = 100 - 9 = 91$

$\Rightarrow mAP^2 = \sqrt{91}$ cm = 9.54cm (approx)

and In $\triangle DPB$.

$m\overline{BD}^2 = m\overline{DP}^2 + m\overline{PB}^2$

$8^2 = 3^2 + m\overline{BP}^2$

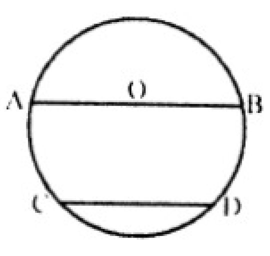
$\Rightarrow m\overline{BP}^2 = 64 - 9 = 55$

$m\overline{BP}^2 = \sqrt{55}$ cm = 7.42cm(approx.)

So, the distance between the centers = $m\overline{AP} + m\overline{BP} = 9.54 + 7.42 = 16.96$ cm

4. Show that greatest chord in a circle is its diameter.

Solution:



Given:

A diameter AB and a chord CD in a circle with centre O.

To prove:

AB > CD

Or greater than any other chord.

Proof:

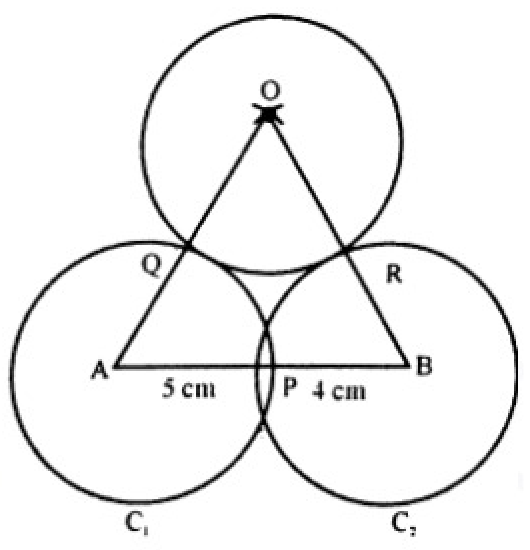
∴ AB is nearer the center the CD.

∴ AB > CD

Hence, AB, being nearest the center then all chords. So, AB is greater than any one of them.

Exercise 10.3

1. Two circles with radii 5cm and 4cm touch each other externally. Draw another circle with radius 2.5cm touching the first pair, externally.



Solution:

Construction:

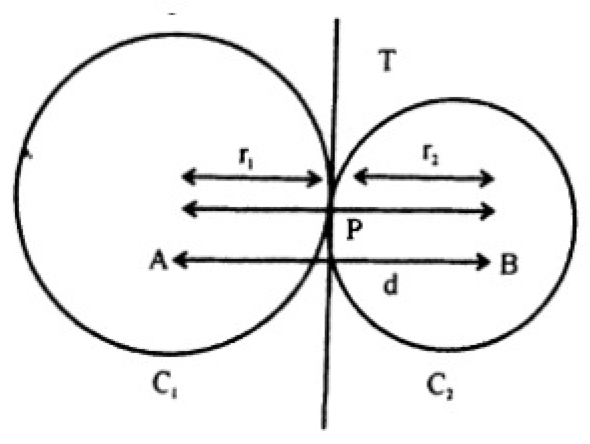
1. Draw two circles C_1 and C_2 having radius 5cm and 4cm touch each other at point P.

2. Draw an arc having radius of 7.5 cm from point. A and another arc from point B having radius 6.5cm cut each other at point O.

3. With a radius of 2.5 draw a circle from point. 'O' which touches the circles C_1 and C_2 at 'Q' and 'R'.

Hence it is required circle.

2. If the distance between the centers of two circles is the sum or the difference of their radii, they will touch each other.



Solution:

Given:

Two circles with centers 'A' and 'B' touch each other at P.

To prove:

$d = r_1 + r_2$

Construction:

AP is the radius and PT, the common tangent at the point P to both the circles.

Proof:

Since AP is the radius at P and PT is tangent at the point 'P' therefore

$\angle APT = 90°$ ________(i)

and $\angle BPT = 90°$ ________(ii)

By adding (i) and (ii), we have

$\angle APT + \angle BPT = 180°$

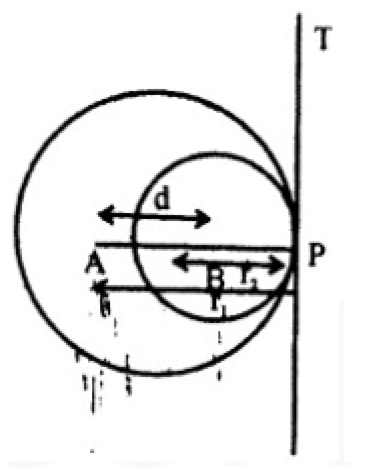
$\Rightarrow$ APB is a straight line.

If r_1 and r_2 are the radii of two circles and d, the distance between the two centers. Then the two circles touch externally.

$d = r_1 + r_2$

Hence proved

(b) To prove, $d = r_1 - r_2$



If r_1 and r_2 are the radii of two circles and d, the distance between the two centers, then the two circles touch internally.

$d = r_1 - r_2$

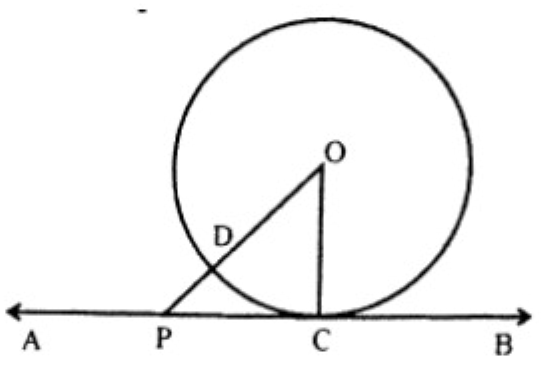
Hence proved

THEOREM 1

10.1 (i) If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.

Given:

A circle with center O and OC is the radial segment. $\overleftrightarrow{AB}$ is perpendicular to $\overline{OC}$ at its outer end C.



To prove:

$\overleftrightarrow{AB}$ is a tangent to the circle at C.

Construction:

Take any point P other than C on $\overleftrightarrow{AB}$. Join O with P.

Proof:

Statements	Reasons
In $\triangle OCP$,	
$m\angle OCP = 90^\circ$	$\overleftrightarrow{AB} \perp \overline{OC}$ (given)
and $m\angle OPC < 90^\circ$	Acute angle of right-angled triangle.
$m\overline{OP} > m\overline{OC}$	Greater angle has greater side opposite to it.
P is a point outside the circle.	$\overline{OC}$ is the radial segment.
Similarly, every point on $\overline{AB}$ except C lies outside the circle.	
Hence $\overleftrightarrow{AB}$ intersects the circle at one-point C only.	

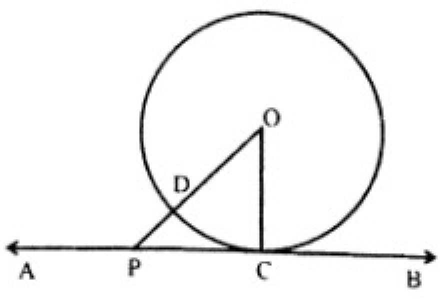
i.e., $\overleftrightarrow{AB}$ is a tangent to the circle at one point only.	

THEOREM 2

10.1 (ii) The tangent to a circle and the radial segment joining the point of contact and the center are perpendicular to each other.

Given:

In a circle with center O and radius $\overline{OC}$, $\overleftrightarrow{AB}$ is the tangent to the circle at point C.



To prove:

Tangent line $\overleftrightarrow{AB}$ and radial-segment $\overline{OC}$ are perpendicular to each other.

Construction:

Take any point P other than C on tangent line $\overleftrightarrow{AB}$.

Join O with P so that $\overline{OP}$ meets the circle at D.

Proof:

Statements	**Reasons**
Line $\overline{AB}$ is the tangent to the circle at point C. Whereas $\overline{OP}$ cuts the circle at D.	Given Construction
$m\,\overline{OC} = m\,\overline{OD}$ (i)	Radii of the same circle
But $m\,\overline{OD} < m\overline{OP}$ (ii)	Point P is outside the circle.
$m\,\overline{OC} < m\overline{OP}$	Using (i) and (ii)
So, radius $\overline{OC}$ is shortest of all lines that can be drawn from O to the tangent line $\overleftrightarrow{AB}$.	

Also $\overline{OC} \perp \overline{AB}$	
Hence, radial segment $\overline{OC}$ is perpendicular to the tangent line $\overleftrightarrow{AB}$.	

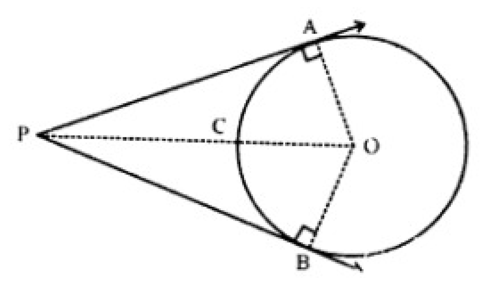
Corollary: There can only be one perpendicular drawn to the radial segment $\overline{OC}$ at the point C of the circle. It follows that one and only one tangent can be drawn to the circle at the given point C on its circumference.

THEOREM 3

10.1 (iii) Two tangents drawn to a circle from a point outside it, are equal in length.

Given:

Two tangents $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are drawn from an external point P to the circle with center O.



To prove:

$m\overline{PA} = m\overline{PB}$

Construction:

Join O with A, O with B and O with P, so that we form ∠rt △ˢ OAP and OBP.

Proof:

Statements	Reasons
In ∠rt △ˢ OAP ↔ OBP	
m∠OAP = m∠OBP = 90°	Radii ⊥ to the tangents PA and PB
hyp. $\overline{OP}$ = hyp. $\overline{OP}$	Common
$m\overline{OA} = m\overline{OB}$	Radii of the same circle.
∴ □OAP ≅□OBP	In ∠rt △ˢ H.S ≅ H.S
Hence, $m\overline{PA} = m\overline{PB}$	

Example 1:

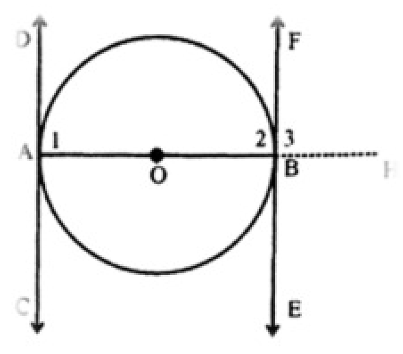
AB is a diameter of a given circle with center O. Tangents are drawn at the end points A and B. Show that the two tangents are parallel.

Given:

$\overline{AB}$ is a diameter of a given circle with center O.

$\overleftrightarrow{CD}$ is the tangent to the circle at point A;

A and $\overline{EF}$ is another tangent at point B.



To prove:

$\overline{CD} \parallel \overline{EF}$

Construction:

Produce diameter $\overline{AB}$ to point H then write ∠1, ∠2 and ∠3 as shown in the Figure.

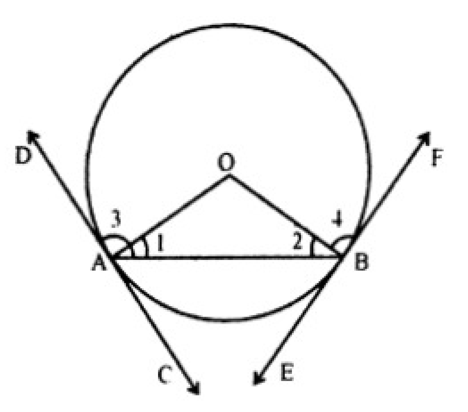
Proof:

Statements	Reasons
$\overline{AB}$ is the diameter of a circle with center O.	Given
$\overline{OA}$ and $\overline{OB}$ are radii of the same circle.	
Moreover $\overleftrightarrow{CD}$ is a tangent to the circle at A.	Given
∴ $\overline{OA} \perp \overline{CD}$	By Theorem 1
⇒ $\overline{AB} \perp \overleftrightarrow{CD}$ (i)	
Similarly $\overleftrightarrow{EF}$ is the tangent to the circle at B	Given
So $\overline{OB} \perp \overleftrightarrow{EF}$	By Theorem 1

⇒ $\overline{AB} \perp \overleftrightarrow{EF}$ (ii)	
Hence $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$	Using (i) and (ii) ($\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ are perpendicular to $\overline{AB}$)

Example 2:

In a circle, the tangents drawn at the ends of a chord, make equal angles with that chord.



Given:

$\overline{AB}$ is the chord of a circle with center O.

CAD is the tangent at point A and EBF is another tangent at point B.

To prove:

m∠BAD = m∠ABF

Construction:

Join O with A and O with B so that we form a △OAB

then write ∠l, ∠2, ∠3 and ∠4 as shown in the Figure.

Proof:

Statements	Reasons
In △OAB	
∴ $m\overline{OA} = m\overline{OB}$	Radii of the same circle.
∴ m∠l = m∠2 (i)	Angles opp. to equal sides of △OAB
Also $\overline{OA} \perp \overline{CD}$	Radius is ⊥ to the tangent line
∴ m∠3 = m∠OAD = 90° (ii)	
Similarly, $\overline{OB} \perp \overline{EF}$	Radius is ⊥ to the tangent
∴ m∠4 = m∠OBF = 90° (iii)	
Hence m∠3 = m∠4 (iv)	Using (ii) and (iii)
⇒ m∠l + m∠3 = m∠2 + m∠4	Adding (i) and (iv)
i.e., m∠BAD = m∠ABF	

THEOREM 4 (A)

10.1 (iv) If two circles touch externally then the distance between their centers is equal to the sum of their radii.

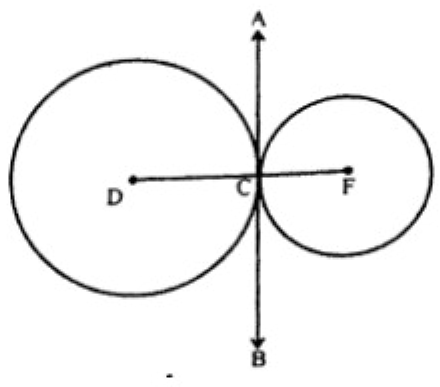
Given:

Two circles with centers D and F respectively touch each other externally at point. So that $\overline{CD}$ and $\overline{CF}$ are respectively the radii of the two circles.

To prove:

(i) Point C lies on the join of centers D and F.

(ii) $m\overline{DF} = m\overline{DC} + m\overline{CF}$



Construction:

Draw $\overline{ACB}$ as a common tangent to the pair of circles at C.

Join C with D and C with F.

Proof:

Statements	Reasons
Both circles touch externally at C whereas $\overline{CD}$ is radial segment and ACB is the common tangent.	
∴ m∠ACD = 90° (i)	Radial segment CD ⊥ the tangent line AB
Similarly CF is radial segment and ACB is the common tangent	
m∠ACF = 90° (ii)	Radial segment CF ⊥ the tangent line AB
m∠ACD + m∠ACF = 90° + 90°	Adding (i) and (ii)

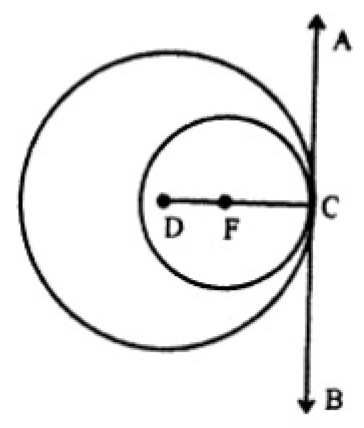
m∠DCF = 180° (iii)	Sum of supplementary adjacent angles
Hence DCF is a straight line with point C between D and F	
so that $m\overline{DF} = m\overline{DC} + m\overline{CF}$	

THEOREM 4 (B)

10.1 (v) If two circles touch each other internally, then the point of contact lies on the line segment through their centers and distance between their centers is equal to the difference of their radii.

Given:

Two circles with centers D and F touch each other internally at point C. So that $\overline{CD}$ and $\overline{CF}$ are the radii of two circles.



To prove:

(i) Point C lies on the join of centers D and F extended,

(ii) $m\overline{DF} = m\overline{DC} - m\overline{CF}$

Construction:

Draw $\overline{ACB}$ as the common tangent to the pair of circles at C.

Proof:

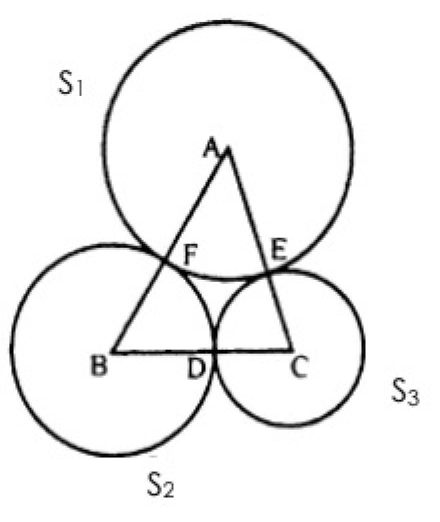
Statements	Reasons
Both circles touch internally at C whereas $\overline{ACB}$ is the common tangent and $\overline{CD}$ is the radial segment of the first circle.	
∴ $m\angle ACD = 90°$ (i)	Radial segment CD ⊥ the tangent line AB
Similarly ACB is the common tangent and CF is the radial segment of the second circle.	
$m\angle ACF = 90°$ (ii)	Radial segment CF ⊥ the tangent line AB.
⇒ $m\angle ACD = m\angle ACF = 90°$	Using (i) and (ii)

Where ∠ACD and ∠ACF coincide each other with point F between D and C.	
Hence $m\overline{DC} = m\overline{DF} + m\overline{FC}$ (iii)	
i.e., $m\overline{DC} - m\overline{FC} = m\overline{DF}$	
or $m\overline{DF} = m\overline{DC} - m\overline{FC}$	

Corollary: If two congruent circles touch each other internally the distance between their centers is equal to zero.

Example 1:

Three circles touch in pairs externally. Prove that the perimeter of a triangle formed by joining centers is equal to the sum of their diameters.



Given:

Three circles have centers A, B and C. Their radii are r_1, r_2 and r_3 respectively. They touch in pairs externally at D, E and F. So that △ ABC is formed by joining the centers of these circles.

To prove:

Perimeter of △ABC = Sum of the diameters of these circles.

Proof:

Statements	Reasons
Three circles with centers A, B and C touch in pairs externally at the points, D, E and F.	Given

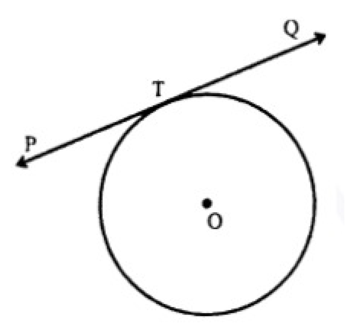
$m\overline{AB} = m\overline{AF} + m\overline{FB}$ (i)	
$m\overline{BC} = m\overline{BD} + m\overline{DC}$ (ii)	
and $m\overline{CA} = m\overline{CE} + m\overline{EA}$ (iii)	
$m\overline{AB} + m\overline{BC} + m\overline{CA} = m\overline{AF} + m\overline{FB} + m\overline{BD} + m\overline{DC} + m\overline{CE} + m\overline{EA}$	Adding (i), (ii) and (iii)
$= (m\overline{AF} + m\overline{EA}) + (m\overline{FB} + m\overline{BD}) + (m\overline{CD} + m\overline{CE})$	
Perimeter of △ABC = $2r_1 + 2r_2 + 2r_3$	$d_1 = 2r_1$, $d_2 = 2r_2$ and $d_3 = 2r_3$ are diameters of the circles.
$= d_1 + d_2 + d_3$	
= Sum of diameters of the circles.	

Miscellaneous Exercise 10

QI. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

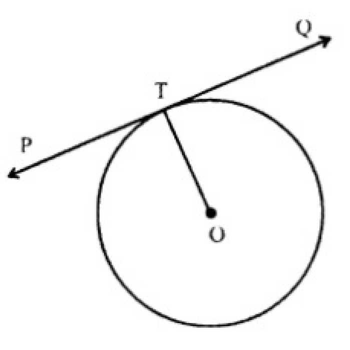
(i) In the adjacent figure of the circle, the line $\overline{PTQ}$ is named as



(a) an arc **(b)** a chord

(c) a tangent **(d)** a secant

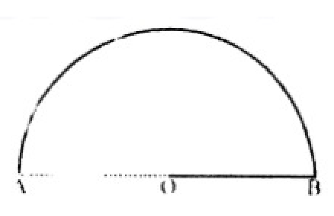
(ii) In a circle with center O, if $\overline{OT}$ is the radial segment and $\overline{PTQ}$ is the tangent line, then



(a) $\overline{OT} \perp \overline{PQ}$ **(b)** $\overline{OT} \not\perp \overline{PQ}$

(c) $\overline{OT} \parallel \overline{PQ}$ **(d)** $\overline{OT}$ is right bisector of $\overline{PQ}$

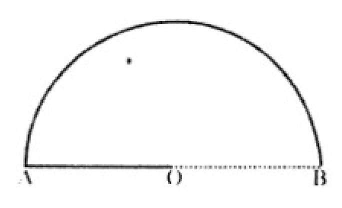
(iii) In the adjacent figure find semicircular area if $\pi = 3.1416$ and $m\overline{OA} = 20$cm.



(a) 62.83sq cm **(b)** 314.16sq cm

(c) 436.20sq cm **(d)** 628.32sq cm

(iv) In the adjacent figure find half the perimeter of circle with center O if π = 3.1416 and $m\overline{OA} = 20$ cm.



(a) 31.42cm **(b)** 62.832cm

(c) 125.65cm **(d)** 188.50 cm

(v) A line which has two points in common with a circle is called.

(a) sine of a circle **(b)** cosine of a circle

(c) tangent of a circle **(d)** secant of a circle

(vi) A line which has only one point in common with a circle is called

(a) sine of a circle **(b)** cosine of a circle

(c) tangent of a circle **(d)** secant of a circle

(vii) Two tangents drawn to a circle from a point outside it are of ..... in length

(a) half **(b)** equal

(c) double **(d)** triple

(viii) A circle has only one

(a) secant **(b)** chord

(c) diameter **(d)** center

(ix) A tangent line intersects the circle at

(a) three points **(b)** two points

(c) single point **(d)** no point at all

(x) Tangents drawn at the ends of diameter of a circle are ...... to each other

(a) parallel **(b)** non parallel

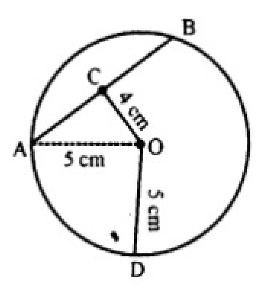
(c) collinear **(d)** perpendicular

(xi) The distance between the centers of two congruent touching circles externally is

(a) of zero length **(b)** the radius of each circle

(c) the diameter of each circle **(d)** twice the diameter of each circle

(xii) In the adjacent circular figure with center O and radius 5cm, the length of the chord intercepted at 4cm away from the center of this circle is:

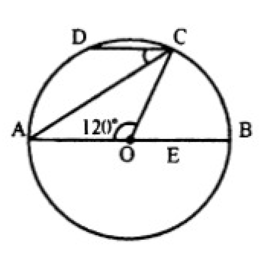


(a) 4 cm **(b)** 6 cm

(c) 7cm **(d)** 9cm

(xiii) In the adjoining figure there is a circle with center O.

If DC || diameter AB and m∠AOC = 120°, then m∠ACD is:



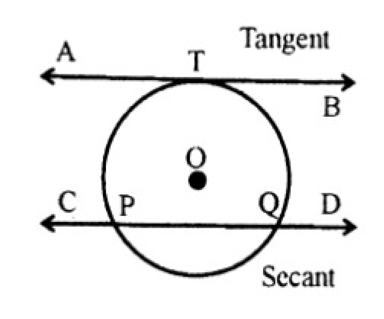
(a) 40° **(b)** 30°

(c) 50° **(d)** 60°

Answers:

(i)	c	**(ii)**	a	**(iii)**	d	**(iv)**	b	**(v)**	d
(vi)	c	**(vii)**	b	**(viii)**	d	**(ix)**	c	**(x)**	a
(xi)	c	**(xii)**	b	**(xiii)**	b				

Summary



✓ A secant is a straight line which-cuts the circumference of a circle in two distinct points. In the figure, the secant CD cuts the circle at two distinct points P and Q.

✓ A tangent to a circle is the straight line which touches the circumference at one point only. The point of tangency is also known as the point of contact. In the figure AB is the tangent line to the circle at the point T.

✓ The length of a tangent to a circle is measured from the given point to the point of contact.

✓ A tangent to a circle is perpendicular to the radial segment drawn to the point of contact.

✓ If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.

✓ The tangent to a circle and the radial segment joining the point of contact and the center are perpendicular to each other.

✓ The two tangents drawn to a circle from a point outside it, are equal in length.

✓ If two circles touch externally or internally, the distance between their centers is respectively equal to the sum or difference of their radii.