Solved Exercise 2.1

1. Find the discriminant of the following given quadratic equation:

$(i)\ 2x^2+3x-1=0$

Solution:

$2x^2+3x-1=0$

Compare it with

$ax^2+bx+c=0$

Here $a=2, b=3, c=-1$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (3)^2-4(2)(-1) \\ &= 9+8 \\ &= 17 \end{aligned}$$

$(ii)\ 6x^2-8x+3=0$

Solution:

$6x^2-8x+3=0$

Compare it with

$ax^2+bx+c=0$

Here $a=6, b=-8, c=3$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (-8)^2-4(6)(3) \\ &= 64-72 \\ &= -8 \end{aligned}$$

$(iii)\ 9x^2-30x+25=0$

Solution:

$9x^2-30x+25=0$

Compare it with

$ax^2+bx+c=0$

Here $a=9, b=-30, c=25$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (-30)^2-4(4)(25) \\ &= 900-900 \\ &= 0 \end{aligned}$$

$(iv)\ 4x^2-7x-2=0$

Solution:

$4x^2-7x-2=0$

Compare it with

$ax^2+bx+c=0$

Here $a=4, b=-7, c=-2$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (-7)^2-4(4)(-2) \\ &= 49+32 \\ &= 81 \end{aligned}$$

2. Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:

$(i)\ x^2-23x+120=0$

Solution :

$x^2-23x+120=0$

Compare it with

$ax^2+bx+c=0$

Here $a=1, b=-23, c=120$

Disc. $=b^2-4ac$

$$=(-23)^2-4(1)(120)$$
$$=529-480$$
$$=49$$
$$=(7)^2>0$$

As the disc. is positive and is a perect square. Therefore the roots are rational (real) and unequal. Verification by solving the equation.

$x^2-23x+120=0$

$x^2-15x-8x+120=0$

$x(x-15)-8(x-15)=0$

$(x-8)(x-15)=0$

Either $x-8=0$ *or* $x-15=0$

$x=8$ $x=15$

Thus, the roots are rational (real) and unequal.

(ii) $2x^2+3x+7=0$

Solution :

$2x^2+3x+7=0$

Compare it with

$ax^2+bx+c=0$

Here $a=2, b=3, c=7$

Disc. $=b^2-4ac$

$$=(3)^2-4(2)(7)$$
$$=9-56$$
$$=-47<0$$

As the disc. is negative

Therefore the roots are imaginary and unequal.

Verification by solving the equation.

$2x^2+3x+7=0$

using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-3\pm\sqrt{(3)^2-4(2)(7)}}{2(2)}$$
$$=\frac{-3\pm\sqrt{9-56}}{4}$$
$$=\frac{-3\pm\sqrt{-47}}{4}$$

Thus, the roots are imaginary and unequal.

(iii) $16x^2-24x+9=0$

Solution :

$16x^2-24x+9=0$

Compare it with

$ax^2+bx+c=0$

Here $a=16, b=-24, c=9$

Disc. $=b^2-4ac$

$$=(-24)^2-4(16)(9)$$

$= 576 - 576$

$= 0$

As the disc. is zero.

Therefore the roots are real and equal.

Verification by solving the equation.

$16x^2 - 24x + 9 = 0$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$= \frac{24 \pm \sqrt{576 - 576}}{32}$$

$$= \frac{24 \pm \sqrt{0}}{32}$$

$$= \frac{24}{32} = \frac{3}{4}$$

Thus, the roots are real and equal.

$(iv)\ 3x^2 + 7x - 13 = 0$

Solution :

$3x^2 + 7x - 13 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 3, b = 7, c = -13$

Disc. $= b^2 - 4ac$

$= (7)^2 - 4(3)(-13)$

$= 49 + 156$

$= 205 > 0$

As the disc. is positive and not a perfect square.

Therefore the roots are irrational $(real)$ and unequal.

Verification by solving the equation.

$3x^2 + 7x - 13 = 0$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$= \frac{-7 \pm \sqrt{205}}{6}$$

Thus, the roots are irrational (real) and unequal.

3. For what value of A, the expression $k^2x^2 + 2(k+1)x + 4$ is perfect square.

Solution :

$k^2x^2+2(k+1)x+4=0$

Compare it with

$ax^2+bx+c=0$

Here $a=k^2,\ b=2(k+1),\ c=4$

Disc. $=b^2-4ac$

$=[2(k+1)]^2-4(k^2)(4)$

$=4(k+1)^2-16k^2$

$=4(k^2+2k+1)-16k^2$

$=4k^2+8k+4-16k^2$

$=-12k^2+8k+4=0$

As the disc. of the given expression is a perfect square.

Therefore the roots are rational and unequal.

So, Disc=0

$-12k^2+8k+4=0$

$-(12k^2-8k-4)=0$

$\Rightarrow\ 12k^2-8k-4=0$

$12k^2-12k+4k-4=0$

$12k(k-1)+4(k-1)=0$

$(12k+4)(k-1)=0$

Either $12k+4=0$ *or* $k-1=0$

$12k=-4$ $\quad k=1$

$k=\dfrac{-4}{12}$

$k=\dfrac{-1}{3}$

4. Find the value of k, if the roots of the following equations are equal.

$(i)\ (2k-1)x^2+3kx+3=0$

Solution :

$(2k-1)x^2+3kx+3=0$

Compare it with

$ax^2+bx+c=0$

Here $a=2k-1,\ b=3k,\ c=3$

As the roots are equal, So

Disc. $=0$

$b^2-4ac=0$

$(3k)^2-4(2k-1)(3)=0$

$9k^2-12(2k-1)=0$

$9k^2-24k+12=0$

$3(3k^2-8k+4)=0$

$\Rightarrow 3k^2-8k+4=0$

$3k^2-6k-2k+4=0$

$3k(k-2)-2(k-2)=0$

$(3k-2)(k-2)=0$

Either $3k-2=0$ *or* $k-2=0$

$3k=2$ $\quad k=2$

$k=\dfrac{2}{3}$

$(ii)\ x^2+2(k+2)x+(3k+4)=0$

Solution :

$x^2+2(k+2)x+(3k+4)=0$

Compare it with

$ax^2+bx+c=0$

Here $a=1,\ b=2(k+2),\ c=3k+4$

As the roots are equal, So

Disc. $=0$

$b^2-4ac=0$

$$[2(k+2)]^2-4(1)(3k+4)=0$$
$$4(k+2)^2-4(3k+4)=0$$
$$4(k^2+4k+4)-12k-16=0$$

$$4k^2+16k+16-12k-16=0$$
$$4k^2+4k=0$$
$$4k(k+1)=0$$
$$Either \quad 4k=0 \quad or \quad k+1=0$$
$$k=0 \qquad\qquad k=-1$$

(iii) $(3k+2)x^2-5(k+1)x+(2k+3)=0$

Solution:

$$(3k+2)x^2-5(k+1)x+(2k+3)=0$$

Compare it with

$$ax^2+bx+c=0$$

Here $a=3k+2,\ b=-5(k+1),\ c=2k+3$

As the roots are equal, So

$$Disc.=0$$
$$b^2-4ac=0$$
$$[-5(k+1)]^2-4(3k+2)(2k+3)=0$$
$$25(k^2+2k+1)-4(6k^2+13k+6)=0$$
$$25k^2+50k+25-24k^2-52k-24=0$$
$$k^2-2k+1=0$$
$$(k-1)^2=0$$
$$k-1=0$$
$$k=1$$

5. Show that the equation $x^2+(mx+c)^2=a^2$ **has equal roots,**

if $c^2=a^2(1+m^2)$

Solution:

$$x^2+(mx+c)^2=a^2$$
$$x^2+m^2x^2+2mcx+c^2=a^2$$
$$(1+m^2)x^2+2mcx+c^2=a^2$$

$$(1+m^2)x^2+2mcx+c^2-a^2=0$$

Here $a=1+m^2,\ b=2mc,\ c=c^2-a^2$

As the roots are equal, So

$$Disc.=0$$
$$b^2-4ac=0$$
$$(2mc)^2-4(1+m^2)(c^2-a^2)=0$$
$$4m^2c^2-4(c^2-a^2+m^2c^2-a^2m^2)=0$$
$$4m^2c^2-4c^2+4a^2-4m^2c^2+4a^2m^2=0$$
$$-4c^2+4a^2+4a^2m^2=0$$
$$-4(c^2-a^2-a^2m^2)=0$$
$$c^2-a^2-a^2m^2=0$$
$$c^2=a^2+a^2m^2$$
$$c^2=a^2(1+m^2)$$

Hence proved.

6. Find the condition that the roots of the equation $(mx+c)^2-4ax=0$ **are equal.**

Solution:

$(mx+c)^2-4ax=0$

$m^2x^2+2mcx+c^2-4ax=0$

$m^2x^2+2mcx-4ax+c^2=0$

$m^2x^2+2(mc-2a)x+c^2=0$

Here $a=m^2,\ b=2(mc-2a),\ c=c^2$

As the roots are equal, So

$Disc.=0$

$b^2-4ac=0$

$[2(mc-2a)]^2-4(m^2)(c^2)=0$

$4(m^2c^2-4amc+4a^2)-4(m^2c^2)=0$

$4(m^2c^2-4amc+4a^2-m^2c^2)=0$

$4(4a^2-4amc)=0$

$\Rightarrow 4a^2-4amc=0$

$4a(a-mc)=0$

$a-mc=0$

$a=mc$

which is the required condition.

7. If the roots of the equation $(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$ **are equal, then** $a=0\ or\ a^3+b^3+c^3=3abc$

Solution:

$(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$

Here $a=c^2-ab,\ b=-2(a^2-bc),\ c=b^2-ac$

As the roots are equal, So

$Disc.=0$

$b^2-4ac=0$

$[-2(a^2-bc)]^2-4(c^2-ab)(b^2-ac)=0$

$4(a^2-bc)^2-4(c^2-ab)(b^2-ac)=0$

$4[(a^4-2a^2bc+b^2c^2)-(b^2c^2-ac^3-ab^3+a^2bc)]=0$

$a^4-2a^2bc+b^2c^2-b^2c^2+ac^3+ab^3-a^2bc=0$

$\Rightarrow a^4+ab^3+ac^3-3a^2bc=0$

$a(a^3+b^3+c^3-3abc)=0$

Either a=0 or $a^3+b^3+c^3-3abc=0$

$a^3+b^3+c^3=3abc$

Hence proved.

8. Show that the roots of the following equations are rational.

$(i)\ a(b-c)x^2+b(c-a)x+c(a-b)=0$

Solution:

$a(b-c)x^2+b(c-a)x+c(a-b)=0$

Here $a=a(b-c),\ b=b(c-a),\ c=c(a-b)$

Disc.

$=b^2-4ac$

$=\left[b(c-a)\right]^2-4\left[a(b-c)\right]\left[c(a-b)\right]$

$=b^2(c-a)^2-4ac(b-c)(a-b)$

$=b^2(c-a)^2-4ac\left(ab-b^2-ac+bc\right)$

$=b^2\left(c^2+a^2-2ac\right)-4ac\left(ab-b^2-ac+bc\right)$

$=b^2c^2+a^2b^2-2ab^2c-4a^2bc+4ab^2c+4a^2c^2-4abc^2$

$=a^2b^2+b^2c^2+4a^2c^2+2ab^2c-4a^2bc-4abc^2$

$=(ab)^2+(bc)^2+(-2ac)^2+2(ab)(bc)+2(bc)(-2ac)+2(-2ac)(ab)$

$=(ab+bc-2ac)^2$

Hence the roots are rational.

$(ii)\ (a+2b)x^2+2(a+b+c)x+(a+2c)=0$

Solution :

$(a+2b)x^2+2(a+b+c)x+(a+2c)=0$

Here $a=a+2b,\ b=2(a+b+c),\ c=a+2c$

Disc.

$=b^2-4ac$

$=\left[2(a+b+c)\right]^2-4(a+2b)(a+2c)$

$=4(a+b+c)^2-4\left(a^2+2ac+2ab+4bc\right)$

$=4\left(a^2+b^2+c^2+2ab+2bc+2ca\right)-4\left(a^2+2ac+2ab+4bc\right)$

$=4\left[a^2+b^2+c^2+2ab+2bc+2ca-a^2-2ac-2ab-4bc\right]$

$=4\left[b^2+c^2-2bc\right]$

$=4(b-c)^2$

$=\left[2(b-c)\right]^2$

Hence the roots are rational.

9. For all values of k, prove that the roots of the equation

$x^2-2\left(k+\frac{1}{k}\right)x+3=0, k\neq 0$ are real.

Solution :

$x^2-2\left(k+\frac{1}{k}\right)x+3=0$

Here $a=1, b=-2\left(k+\frac{1}{k}\right), c=3$

Disc.

$=b^2-4ac$

$=\left[-2\left(k+\frac{1}{k}\right)\right]^2-4(1)(3)$

$=4\left(k+\frac{1}{k}\right)^2-12$

$=4\left[\left(k+\frac{1}{k}\right)^2-3\right]$

$=4\left[k^2+\frac{1}{k^2}+2-3\right]$

$=4\left[k^2+\frac{1}{k^2}-1\right]>0$

Hence, the roots are real.

10. Show that the roots of the equation

$(b-c)x^2+(c-a)x+(a-b)=0$ **are real.**

Solution :

$(b-c)x^2+(c-a)x+(a-b)=0$

Here $a=(b-c), b=(c-a), c=(a-b)$

Disc.

$$= b^2 - 4ac$$
$$= (c-a)^2 - 4(b-c)(a-b)$$
$$= c^2 + a^2 - 2ac - 4\left(ab - b^2 - ac + bc\right)$$
$$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$$

$$= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac$$
$$= (a)^2 + (-2b)^2 + (c)^2 + 2(a)(-2b) + 2(-2b)(c) + 2(a)(c)$$
$$= (a - 2b + c)^2 > 0$$

Hence the roots of equation are real.

Cube roots of unity and their properties

The cube roots of unity:

Let a number x be the cube root of unity,

i.e. $x=(1)^{\frac{1}{3}}$

Solution:

$$
\begin{aligned}
or\quad & x^3=1\\
\Rightarrow\quad & x^3-1=0\\
& (x)^3-(1)^3=0\\
& (x-1)(x^2+x+1)=0 \qquad \left[\text{using } a^3-b^3=(a-b)(a^2+ab+b^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
Either\ \ x-1=0 \quad & or \quad x^2+x+1=0\\
\Rightarrow\ \ x=1 \quad & or \quad x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}\\
& =\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}
\end{aligned}
$$

Three cube roots of unity are

$1, \frac{-1+i\sqrt{3}}{2}$ *and* $\frac{-1-i\sqrt{3}}{2}$, where $i=\sqrt{-1}$

Recognize complex cube roots of unity as ω and ω^2:

The two complex cube roots of unity are $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

If we name anyone of these as w (pronounced as omega), then the other is ω^2

Properties of cube roots of unity:

(a) Prove that each of the complex cube roots of unity is the square of the other.

Proof:

The complex cube roots of unity are $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

We prove that

Solution:

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{-1-\sqrt{-3}}{2}, \qquad \text{and} \qquad \left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{-1+\sqrt{-3}}{2}$$

$$
\begin{aligned}
\left(\frac{-1+\sqrt{-3}}{2}\right)^2 &=\frac{1+(-3)-2\sqrt{-3}}{4} & \left(\frac{-1-\sqrt{-3}}{2}\right)^2 &=\frac{-1+(-3)+2\sqrt{-3}}{4}\\
&=\frac{-2-2\sqrt{-3}}{4} & &=\frac{-2+2\sqrt{-3}}{4}\\
&=\frac{2(-1-\sqrt{-3})}{4} & &=\frac{2(-1+\sqrt{-3})}{4}\\
&=\frac{-1-\sqrt{-3}}{2} & &=\frac{-1+\sqrt{-3}}{2}
\end{aligned}
$$

Thus, each of the complex cube root of unity is the square of the other, that is,

if $\omega=\frac{-1+\sqrt{-3}}{2}$, *then* $\omega^2=\frac{-1-\sqrt{-3}}{2}$ and if $\omega=\frac{-1-\sqrt{-3}}{2}$ *then*

$\omega^2=\frac{-1+\sqrt{-3}}{2}$

(b) Prove that the product of three cube roots of unity is one.

Proof:

Solution:

Three cube roots of unity are

$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$

The product of cube roots of unity $=(1)\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$

$$=\frac{(-1)^2-\left(\sqrt{-3}\right)^2}{4}=\frac{-1-(-3)}{4}=\frac{1+3}{4}=\frac{4}{4}=1$$

i.e., $(1)(\omega)(\omega^2)=1 \text{ or } \omega^3=1$

(c) Prove that each complex cube root of unity is reciprocal of the other.

Proof:

We know that $\omega^3=1 \quad \Rightarrow \quad \omega.\omega^2=1, so$

$$\omega=\frac{1}{\omega^2} \quad or \quad \omega^2=\frac{1}{\omega}$$

Thus, each complex cube root of unity is reciprocal of the other.

(b) Prove that the sum of all the cube roots of unity is zero.

i.e., $1+\omega+\omega^2=0$

Proof:

The cube roots of unity are

$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$

$if\ \omega=\frac{-1+\sqrt{-3}}{2}, then\ \omega^2=\frac{-1-\sqrt{-3}}{2}$

The sum o all the roots $=1+\omega+\omega^2$

$$=1+\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$

$$=\frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2}=\frac{0}{2}=0$$

Thus $1+\omega+\omega^2=0$

We can easily deduce the following results, that is

$(i) \quad 1+\omega^2=-\omega \quad (ii) \quad 1+\omega=-\omega^2 \quad (iii) \quad \omega+\omega^2=-1$

Exercise 2.2

1. Find the cube roots of -1,8, -27,64.

(i) The three cube roots of -1

Solution:

Let $x^3=-1$

$(x)^3+(1)^3=0$

$(x+1)(x^2-x+1)=0$

$Either\ x+1=0 \quad or \quad x^2-x+1=0$

$x=-1 \quad Here\ a=1, b=-1, c=1$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$$x=\frac{1\pm\sqrt{1-4}}{2}$$

$$x=\frac{1\pm\sqrt{-3}}{2}$$

$$x=\frac{1+\sqrt{-3}}{2} \quad or \quad x=\frac{1-\sqrt{-3}}{2}$$

$$=-\left(\frac{-1-\sqrt{-3}}{2}\right) \quad =-\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$=-\omega^2 \quad =-\omega$$

Three cube roots of -1 are $-1,-\omega,-\omega^2$

(ii) The three cube roots of 8

Solution:

$\text{Let } x^3=8$

$x^3-8=0$

$(x)^3-(2)^3=0$

$(x-2)(x^2+2x+4)=0$

$$Either \ x-2=0 \qquad or \qquad x^2+2x+4=0$$

$$x=2 \qquad Here \ a=1, b=2, c=4$$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}$$

$$x=\frac{-2\pm\sqrt{4-16}}{2}$$

$$x=\frac{-2\pm\sqrt{-12}}{2}$$

$$x=\frac{-2\pm 2\sqrt{-3}}{2}$$

$$x=\frac{2\left(-1\pm i\sqrt{3}\right)}{2} \because i=\sqrt{-1}$$

$$x=2\left(\frac{-1+i\sqrt{3}}{2}\right) \quad or \quad x=2\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=2\omega \qquad =2\omega^2$$

Three cube roots of 8 are $2, 2\omega, 2\omega^2$

(iii) The three cube roots of -27

Solution:

$\text{Let } x^3=-27$

$x^3+27=0$

$(x)^3+(3)^3=0$

$(x+3)(x^2-3x+9)=0$

$$Either \ x+3=0 \qquad or \qquad x^2-3x+9=0$$

$x=-3$ $\quad\quad$ $Here\ a=1, b=-3, c=9$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}$$
$$x=\frac{3\pm\sqrt{9-36}}{2}$$
$$x=\frac{3\pm\sqrt{-27}}{2}$$
$$x=\frac{3\pm3\sqrt{-3}}{2}$$
$$x=\frac{3\left(1\pm i\sqrt{3}\right)}{2}\because i=\sqrt{-1}$$
$$x=3\left(\frac{1+i\sqrt{3}}{2}\right)\ or\ x-3\left(\frac{1-i\sqrt{3}}{2}\right)$$
$$x=-3\left(\frac{-1-i\sqrt{3}}{2}\right)\ or\ x=-3\left(\frac{-1+i\sqrt{3}}{2}\right)$$
$$=-3\omega^2 \qquad\qquad =-3\omega$$

Three cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) The three cube roots of 64

Solution:

$\text{Let } x^3=64$

$x^3-64=0$

$(x)^3-(4)^3=0$

$(x-4)\left(x^2+4x+16\right)=0$

$Either\ x-4=0$ $\quad or\quad$ $x^2+4x+16=0$

$x=4$ $\quad\quad$ $Here\ a=1, b=4, c=16$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(4)\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$
$$x=\frac{-4\pm\sqrt{16-64}}{2}$$
$$x=\frac{-4\pm\sqrt{-48}}{2}$$
$$x=\frac{-4\pm4\sqrt{-3}}{2}$$
$$x=\frac{4\left(-1\pm i\sqrt{3}\right)}{2}\because i=\sqrt{-1}$$
$$x=4\left(\frac{-1+i\sqrt{3}}{2}\right)\ or\ x=4\left(\frac{-1-i\sqrt{3}}{2}\right)$$
$$=4\omega \qquad\qquad =4\omega^2$$

Three cube roots of 64 are $4, 4\omega, 4\omega^2$

2. Evaluate

(i) $\left(1-\omega-\omega^2\right)^7$

Solution:

$$\begin{aligned}(1-\omega-\omega^2)^7 &= \left[1-(\omega+\omega^2)\right]^7\\ &= \left[1-(-1)\right]^7 \quad \because \omega+\omega^2=-1\\ &= (1+1)^7\\ &= 2^7 = 128\end{aligned}$$

(ii) $\left(1-3\omega-3\omega^2\right)^5$

Solution:

$$\begin{aligned}(1-3\omega-3\omega^2)^5 &= \left[1-3(\omega+\omega^2)\right]^5\\ &= \left[1-3(-1)\right]^5 \because \omega+\omega^2=-1\\ &= (1+3)^5\\ &= 4^5 = 1024\end{aligned}$$

(iii) $\left(9+4\omega+4\omega^2\right)^3$

Solution:

$$\begin{aligned}(9+4\omega+4\omega^2)^3 &= \left[9+4(\omega+\omega^2)\right]^3\\ &= \left[9+4(-1)\right]^3 \because \omega+\omega^2=-1\\ &= (9-4)^3\\ &= 5^3 = 125\end{aligned}$$

(iv) $\left(2+2\omega-2\omega^2\right)\left(3-3\omega+3\omega^2\right)$

Solution:

$$\begin{aligned}&\left(2+2\omega-2\omega^2\right)\left(3-3\omega+3\omega^2\right)\\ &= \left[2(1+\omega)-2\omega^2\right]\left[3(1+\omega^2)-3\omega\right]\\ &= \left[2(-\omega^2)-2\omega^2\right]\left[3(-\omega)-3\omega\right] \quad \because 1+\omega=-\omega^2\\ &= \left(-2\omega^2-2\omega^2\right)(-3\omega-3\omega)\end{aligned}$$

$$\begin{aligned}&= \left(-4\omega^2\right)(-6\omega)\\ &= (-4)(-6)\left(\omega^2.\omega\right)\\ &= 24\omega^3\\ &= 24(1) \quad \because \omega^3=1\\ &= 24\end{aligned}$$

(v) $\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$

Solution:

$$\begin{aligned}&\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6\\ &= \left(2\omega^6\right)+\left(2\omega^2\right)^6 \quad \because \omega=\frac{-1+\sqrt{-3}}{2} \ \textit{and} \ \omega^2=\frac{-1-1\sqrt{-3}}{2}\\ &= 2^6\left(\omega^6\right)+2^6\left(\omega^{12}\right) \quad 2\omega=-1+\sqrt{-3} \ \textit{and} \ 2\omega^2=-1-1\sqrt{-3}\\ &= 2^6\left[\left(\omega^3\right)^2\right]+2^6\left[\left(\omega^3\right)^4\right]\end{aligned}$$

$$=2^6\left[(1)^2\right]+2^6\left[(1)^4\right] \qquad \because \omega^3=1$$
$$=2^6[1+1]$$
$$=2^6.2=2^{6+1}=2^7$$
$$=128$$

(vi) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$

Solution:

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$$

$$=\omega^9+\left(\omega^2\right)^9 \qquad \because \omega=\frac{-1+\sqrt{-3}}{2} \text{ and } \omega^2=\frac{-1-1\sqrt{-3}}{2}$$
$$=\omega^9+\omega^{18}$$
$$=\left(\omega^3\right)^3+\left(\omega^3\right)^6$$
$$=(1)^3+(1)^6 \qquad \because \omega^3=1$$
$$=1+1=2$$

(vii) $\omega^{37}+\omega^{38}-5$

Solution:

$$\omega^{37}+\omega^{38}-5$$
$$=\omega^{37}+\omega^{38}-5$$
$$=\omega^{36}.\omega+\omega^{36}.\omega^2-5$$
$$=\left(\omega^3\right)^{12}.\omega+\left(\omega^3\right)^{12}.\omega^2-5$$
$$=(1)^{12}.\omega+(1)^{12}.\omega^2-5 \qquad \because \omega^3=1$$
$$=\omega+\omega^2-5$$
$$=-1-5 \qquad \because \omega+\omega^2=-1$$
$$=-6$$

(viii) $\omega^{-13}+\omega^{-17}$

Solution:

$$\omega^{-13}+\omega^{-17}$$
$$=\omega^{-13}+\omega^{-17}$$
$$=\omega^{-12-1}+\omega^{-15-2}$$
$$=\omega^{-12}.\omega^{-1}+\omega^{-15}.\omega^{-2}$$
$$=\left(\omega^3\right)^{-4}.\omega^{-1}+\left(\omega^3\right)^{-5}.\omega^{-2}$$

$$=(1)^{-4}.\omega^{-1}+(1)^{-5}.\omega^{-2} \qquad \because \omega^3=1$$
$$=\omega^{-1}+\omega^{-2}$$
$$=\frac{1}{\omega}+\frac{1}{\omega^2}$$

$$=\frac{\omega^2+\omega}{\omega^3} \qquad \because \omega+\omega^2=-1 \text{ and } \omega^3=1$$
$$=\frac{-1}{1}$$
$$=-1$$

3. Prove that $x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$.

Solution:

$$x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$$

$$\begin{aligned}
R.H.S &=(x+y)(x+\omega y)(x+\omega^2 y)\\
&=(x+y)\left[x\left(x+\omega^2 y\right)+\omega y\left(\omega+\omega^2 y\right)\right]\\
&=(x+y)\left[x^2+\omega^2 xy+\omega xy+\omega^3 y^2\right]\\
&=(x+y)\left[x^2+\left(\omega^2+\omega\right)xy+(1)y^2\right] \quad \because \omega^3=1\\
&=(x+y)\left[x^2+(-1)xy+y^2\right] \qquad \because \omega^2+\omega=-1\\
&=(x+y)\left(x^2-xy+y^2\right)\\
&=x^3+y^3\\
&=L.H.S
\end{aligned}$$

Hence Proved.

4. Prove that $x^3+y^3+z^3-3xyz=(x+y+z)\left(x+\omega y+\omega^2 z\right)\left(x+\omega^2 y+\omega z\right)$.

Solution:

$$x^3+y^3+z^3-3xyz=(x+y+z)\left(x+\omega y+\omega^2 z\right)\left(x+\omega^2 y+\omega z\right)$$

$$\begin{aligned}
R.H.S &=(x+y+z)\left(x+\omega y+\omega^2 z\right)\left(x+\omega^2 y+\omega z\right)\\
&=(x+y+z)\left[x\left(x+\omega^2 y+\omega z\right)+\omega y\left(x+\omega^2 y+\omega z\right)+\omega^2 z\left(x+\omega^2 y+\omega z\right)\right]\\
&=(x+y+z)\left[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+\omega^2 xz+\omega^4 yz+\omega^3 z^2\right]\\
&=(x+y+z)\left[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+\omega^2 xz+\omega^3.\omega yz+\omega^3 z^2\right]\\
&=(x+y+z)\left[x^2+\omega^2 xy+\omega xy+\omega^2 yz+(1)\omega yz+\omega^2 xz+\omega xz+(1)y^2+(1)z^2\right] \qquad \because \omega^3=1\\
&=(x+y+z)\left[x^2+\left(\omega^2+\omega\right)xy+\left(\omega^2+\omega\right)yz+\left(\omega^2+\omega\right)xz+y^2+z^2\right]\\
&=(x+y+z)\left[x^2+(-1)xy+(-1)yz+(-1)xz+y^2+z^2\right] \qquad \because \omega^2+\omega=-1
\end{aligned}$$

$$\begin{aligned}
&=(x+y+z)\left[x^2+y^2+z^2-xy-yz-zx\right]\\
&=x^3+y^3+z^3-3xz\\
&=L.H.S
\end{aligned}$$

Hence Proved.

5. Prove that $(1+\omega)\left(1+\omega^2\right)\left(1+\omega^4\right)\left(1+\omega^8\right)$.....2n factors=1.

Solution:

$$\begin{aligned}
L.H.S &=(1+\omega)\left(1+\omega^2\right)\left(1+\omega^4\right)\left(1+\omega^8\right)\text{.....2n factors}\\
&=(1+\omega)\left(1+\omega^2\right)\left(1+\omega^3.\omega\right)\left(1+\omega^2.\omega^6\right)\text{.....2n factors}\\
&=(1+\omega)\left(1+\omega^2\right)\left(1+\omega^3.\omega\right)\left(1+\omega^2.\left(\omega^3\right)^2\right)\text{.....2n factors}\\
&=(1+\omega)\left(1+\omega^2\right)\left(1+(1)\omega\right)\left(1+\omega^2(1)^2\right)\text{.....2n factors} \because \omega^3=1\\
&=(1+\omega)\left(1+\omega^2\right)(1+\omega)\left(1+\omega^2\right)\text{.....2n factors}\\
&=\left(-\omega^2\right)\left(1+\omega^2\right)\left(-\omega^2\right)\left(1+\omega^2\right)\text{.....2n factors} \qquad \because 1+\omega=-\omega^2
\end{aligned}$$

$$=\left[(-\omega^2)(-\omega)\right]\left[(-\omega^2)(-\omega)\right]\text{.....n factors} \qquad \because 1+\omega^2=-\omega$$

$$=\left[\omega^3\right]\left[\omega^3\right]\text{.....n factors}$$

$$=(1)(1)\text{.....n factors} \qquad \because \omega^3=1$$

$$=(1)^n$$

$$=1$$

$$=R.H.S$$

Hence Proved.

Roots and co-efficient of a quadratic equation:

We know that $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ are roots of the equation $ax^2+bx+c=0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

Relation between roots and co-efficient of a quadratic equation:

If $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

then we can find the sum and the product of the roots as follows.

Sum of the roots $=\alpha+\beta$

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

Product of the roots $=\alpha\beta$

$$=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}=\frac{b^2-\left(b^2-4ac\right)}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

If we denote the sum of roots and product of roots by S and P respectively,then

$S=-\frac{b}{a}=-\frac{\text{Co-efficient of x}}{\text{Co-efficient of }x^2}$ *and* $P=\frac{c}{a}=\frac{\text{Constant term}}{\text{Co-efficient of }x^2}$

Exercise 2.3

1. Without solving, find the sum and the product of the following quadratic equations.

(i) $x^2-5x+3=0$

Solution:

$x^2-5x+3=0$

Here a=1, b=-5, c=3

Let α and β be the roots of the given equation

Then Sum of roots $=\alpha+\beta=-\frac{b}{a}=-\frac{(-5)}{1}=5$

And product of roots $=\alpha\beta=\frac{c}{a}=\frac{3}{1}=3$

(ii) $3x^2+7x-11=0$

Solution:

$3x^2+7x-11=0$

Here a=3, b=7, c=-11

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{7}{3}$

And product of roots = $\alpha\beta=\frac{c}{a}=-\frac{11}{3}$

(iii) $px^2-qx+r=0$

Solution:

$px^2-qx+r=0$

Here a=p, b=-q, c=r

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-q)}{p}=\frac{q}{p}$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{r}{p}$

(iv) $(a+b)x^2-ax+b=0$

Solution:

$(a+b)x^2-ax+b=0$

Here a=$a+b$, b=-a, c=b

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-a)}{a+b}=\frac{a}{a+b}$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{b}{a+b}$

(v) $(l+\mathrm{m})x^2+(m+n)x+n-l=0$

Solution:

$(l+\mathrm{m})x^2+(m+n)x+n-l=0$

Here a=$l+\mathrm{m}$, b=$m+n$, c=$n-l$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{m+n}{l+\mathrm{m}}$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{n-l}{l+\mathrm{m}}$

(vi) $7x^2-5mx+9n=0$

Solution:

$7x^2-5mx+9n=0$

Here a=7, b=$-5m$, c=9n

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-5m)}{7}=\frac{5m}{7}$

And product of roots $=\alpha\beta=\frac{c}{a}=\frac{9n}{7}$

2. Find the value of k, if

(i) Sum of the roots of the equation 2kx²-3x+4k=0 is twice the product of the roots.

Solution:

$2kx^2-3x+4k=0$
Here a=2k, b=-3, c=4k

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-3)}{2k}=\frac{3}{2k}$

And product of roots $=\alpha\beta=\frac{c}{a}=\frac{4k}{2k}=2$

As sum of the roots is twice the product of the roots, so

$$\alpha+\beta=2\alpha\beta$$
$$\frac{3}{2k}=2(2)$$
$$\frac{3}{2k}=4$$
$$or\, k=\frac{3}{8}$$

(ii) Sum of the roots of the equation $x^2+(3k-7)x+5k=0$ is $\frac{3}{2}$ **times the product of the roots.**

Solution:

$x^2+(3k-7)x+5k=0$
Here a=1, b=$3k-7$, c=5k

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(3k-7)}{1}=-3k+7$

And product of roots $=\alpha\beta=\frac{c}{a}=\frac{5k}{1}=5k$

As sum of the roots is $\frac{3}{2}$ of the product of the roots, so

$$\alpha+\beta=\frac{3}{2}\alpha\beta$$
$$-3k+7=\frac{3}{2}(5k)$$
$$-3k+7=\frac{15k}{2}$$
$$-6k+14=15k$$
$$15k+6k=14$$
$$21k=14$$

$$k=\frac{14}{21}$$
$$k=\frac{2}{3}$$

3. Find k, if

(i) Sum of the squares of the roots of the equation $4kx^2+3kx-8=0$ is 2.

Solution:

$4kx^2+3kx-8=0$
Here a=4k, b=$3k$, c=-8

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{3k}{4k}=-\frac{3}{4}$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{-8}{4k}$

As sum of the square of roots is twice the product of the roots, so

$$\alpha^2+\beta^2=2$$
$$(\alpha+\beta)^2-2\alpha\beta=2 \qquad \because(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$
$$\left(-\frac{3}{4}\right)^2-2\left(\frac{-8}{4k}\right)=2 \qquad \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$\frac{9}{16}+\frac{16}{4k}=2$$
$$\frac{16}{4k}=2-\frac{9}{16} \quad\rightarrow\quad \frac{16}{4k}=\frac{32-9}{16}$$
$$\frac{16}{4k}=\frac{23}{16} \quad\rightarrow\quad 23\times 4k=16\times 16$$
$$k=\frac{16\times 16}{23\times 4} \quad\rightarrow\quad k=\frac{64}{23}$$

(ii) Sum of the squares of the roots of the equation x²-2kx+(2k+1) =0 is 6.

Solution:

$x^2-2kx+(2k+1)=0$
Here a=1, b=-2k, c=2k+1

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-2k)}{1}=2k$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{2k+1}{1}=2k+1$

As sum of the square of roots is 6 to the product of the roots, so

$$\alpha^2+\beta^2=6$$
$$(\alpha+\beta)^2-2\alpha\beta=6 \qquad \because(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$
$$(2k)^2-2(2k+1)=6 \qquad \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$4k^2-4k-2=6$$
$$4k^2-4k-2-6=0$$
$$4k^2-4k-8=0$$
$$4\left(k^2-k-2\right)=0$$
$$\Rightarrow k^2-k-2=0$$
$$k^2-2k+k-2=0$$
$$k(k-2)+1(k-2)=0$$
$$(k-1)(k-2)=0$$

Either $k+1=0$ *or* $k-2=0$
$k=-1$ $\qquad k=2$

4. Find p, if

(i) The roots of the equation $x^2 - x + p^2 = 0$ differ by unity.

Solution:

$x^2 - x + p^2 = 0$

Here a=1, b=-1, c=p^2

Let α and α-1 be the roots of the given equation

$$\text{Then } \alpha+\alpha\text{-}1 = -\frac{b}{a} \quad \textit{and} \quad \alpha(\alpha\text{-}1) = \frac{c}{a}$$

$$2\alpha - 1 = -\frac{(-1)}{1} \qquad \alpha^2\text{-}1 = \frac{p^2}{1}$$

$$2\alpha - 1 = 1 \qquad \alpha^2\text{-}1 = p$$

$$2\alpha = 1+1 \qquad \textit{put } \alpha = 1 \text{ in above eq.,we get}$$

$$2\alpha = 2 \qquad (1)^2 - 1 = p$$

$$\alpha = 1 \qquad 1 - 1 = p$$

$$p = 0$$

(ii) The roots of the equation x² +3x+p-2=0 differ by 2.

Solution:

$x^2 + 3x + p - 2 = 0$

Here a=1, b=3, c=$p-2$

Let α and $\alpha - 2$ be the roots of the given equation

$$\text{Then } \alpha+\alpha-2 = -\frac{b}{a} \quad \textit{and} \quad \alpha(\alpha-2) = \frac{c}{a}$$

$$2\alpha - 2 = -\frac{3}{1} \qquad \alpha^2 - 2\alpha = \frac{p-2}{1}$$

$$2\alpha - 2 = -3 \qquad \alpha^2 - 2\alpha = p - 2$$

$$2\alpha = -3+2 \qquad \textit{put } \alpha = -\frac{1}{2} \text{ in above eq.,we get}$$

$$\alpha = -\frac{1}{2} \qquad \left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) = p - 2$$

$$\frac{1}{4} + 1 = p - 2$$

$$\frac{1}{4} = p - 2 - 1$$

$$\frac{1}{4} = p - 3$$

$$4p - 12 = 1$$

$$4p = 1 + 12$$

$$4p = 13$$

$$p = \frac{13}{4}$$

5. Find m, if

(i) The roots of the equation x² -7x+3m-5=0 satisfy the relation $3\alpha + 2\beta = 4$.

Solution:

$x^2 - 7x + 3m - 5 = 0$

Here a=1, b=-7, c=$3m-5$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-7)}{1}=7$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{3m-5}{1}=3m-5$

Now $\alpha+\beta=7$

$\beta=7-\alpha$ ....(i) *and* $\alpha\beta=3m-5$ ....(ii)

Since $3\alpha+2\beta=4$ ....(iii)

Put β in eq(iii), we have

$3\alpha+2(7-\alpha)=4$

$3\alpha+14-2\alpha=4$

$3\alpha-2\alpha+14=4$

$\alpha=4-14$

$\alpha=-10$

Put $\alpha=-10$ in eq.(i), we get

$\beta=7+10$

$\beta=17$

Put $\alpha=-10, \beta=17$ in eq.(ii), we get

$(-10)(17)=3m-5$

$5-170=3m$

or $3m=-165$

$m=-55$

(ii) The roots of the equation x² +7x+3m-5=0 satisfy the relation $3\alpha-2\beta=4$.

Solution:

$x^2+7x+3m-5=0$

Here a=1, b=7, c=$3m-5$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(7)}{1}=-\frac{7}{1}=-7$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{3m-5}{1}=3m-5$

Now $\alpha+\beta=-7$

$\beta=-7-\alpha$ ....(i) *and* $\alpha\beta=3m-5$ ....(ii)

Since $3\alpha-2\beta=4$ ....(iii)

Put β in eq(iii), we have

$3\alpha-2(-7-\alpha)=4$

$3\alpha+14+2\alpha=4$

$3\alpha+2\alpha=4-14$

$5\alpha=-10$

$\alpha=-2$

Put $\alpha=-2$ in eq.(i), we get

$$\beta = -7-(-2)$$
$$\beta = -7+2$$
$$\beta = -5$$

Put $\alpha = -2$ *and* $\beta = -5$ in eq.(iii), we get

$$(-2)(-5) = 3m-5$$
$$10 = 3m-5$$

or

$$3m = 10+5$$
$$3m = 15$$
$$m = 5$$

(iii) The roots of the equation 3x² -2x+7m+2=0 satisfy the relation $7\alpha - 3\beta = 18$.

Solution:

$3x^2 - 2x + 7m + 2 = 0$

Here a=3, b=-2, c=$7m+2$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$\alpha+\beta = -\frac{(-2)}{3} \qquad \alpha\beta = \frac{7m+2}{3} \quad \text{........}(ii)$$

$$\alpha+\beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - \alpha \quad \text{........}(i)$$

Since $7\alpha - 3\beta = 18$ ....(iii)

Put $\beta = \frac{2}{3} - \alpha$ in eq(iii), we have

$$7\alpha - 3\left(\frac{2}{3} - \alpha\right) = 18$$
$$7\alpha - 2 + 3\alpha = 18$$
$$7\alpha + 3\alpha = 18 + 2$$

$$10\alpha = 20$$
$$\alpha = 2$$

Put $\alpha = 2$ in eq.(i), we get

$$\beta = \frac{2}{3} - 2$$
$$\beta = \frac{2-6}{3}$$
$$\beta = -\frac{4}{3}$$

Put $\alpha = 2$ *and* $\beta = -\frac{4}{3}$ in eq.(ii), we get

$$(2)\left(-\frac{4}{3}\right) = \frac{7m+2}{3}$$
$$-\frac{8}{3} = \frac{7m+2}{3}$$
$$-\frac{8}{3} \times 3 = 7m+2$$
$$-8 = 7m+2$$
$$7m = -8-2$$

or

$$7m = -10$$
$$m = -\frac{10}{7}$$

6. Find m, if sum and product of the roots of the following equations is equal to a given number λ.

(i) $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Solution:

$(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Here a=2m+3, b=$7m-5$, c=$3m-10$

Let α and β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

$$\alpha+\beta=-\frac{(7m-5)}{2m+3} \qquad \alpha\beta=\frac{3m-10}{2m+3} \;\text{........}(ii)$$

As $\alpha+\beta=\alpha\beta=\lambda$

So $\lambda=-\frac{(7m-5)}{2m+3}$ ....(i) and $\lambda=\frac{3m-10}{2m+3}$ ....(ii)

Comparing eq.(i) and eq(ii), we get

$$-\frac{(7m-5)}{2m+3}=\frac{3m-10}{2m+3}$$
$$-(7m-5)=\frac{3m-10}{2m+3}\times(2m+3)$$
$$-7m+5=3m-10$$
$$3m+7m=5+10$$
$$10m=15$$
$$m=\frac{15}{10}$$
$$m=\frac{3}{2}$$

(ii) $4x^2-(3+5m)x-(9m-17)=0$

Solution:

$4x^2-(3+5m)x-(9m-17)=0$

Here a=4, b=$-(3+5m)$, c=$-(9m-17)$

Let α and β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

$$\alpha+\beta=-\frac{[-(3+5m)]}{4} \qquad \alpha\beta=\frac{-(9m-17)}{4} \;\text{........}(ii)$$
$$\alpha+\beta=\frac{3+5m}{4}$$

As $\alpha+\beta=\alpha\beta=\lambda$

So $\lambda=\frac{3+5m}{4}$ ....(i) and $\lambda=\frac{-(9m-17)}{4}$ ....(ii)

Comparing eq.(i) and eq(ii), we get

$$\frac{3+5m}{4}=\frac{-(9m-17)}{4}$$
$$4(3+5m)=-4(9m-17)$$
$$3+5m=-(9m-17)$$
$$3+5m=-9m+17$$
$$5m+9m=17-3$$
$$14m=14$$
$$\Rightarrow \quad m=1$$

Symmetric functions of the roots of a quadratic equation:

Define symmetric functions of the roots of a quadratic equation:

Definition:

Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.

For example, if

$$\begin{aligned} f(\alpha,\beta) &= \alpha^2+\beta^2, then \\ f(\beta,\alpha) &= \beta^2+\alpha^2 = \alpha^2+\beta^2 \qquad (\because \beta^2+\alpha^2=\alpha^2+\beta^2) \\ &= f(\alpha,\beta) \end{aligned}$$

Exercise 2.4

1. If α, β are the roots of the equation x²+px+q=0, then evaluate

(i) $\alpha^2+\beta^2$

Solution:

Solution:

$\alpha^2+\beta^2$

$x^2+px+q=0$

Here a=1, b=p, c=q

Let α, β be the roots of the given equation

Then $\alpha+\beta = -\dfrac{b}{a}$ *and* $\alpha\beta = \dfrac{c}{a}$

$= -\dfrac{p}{1}$ $\qquad \alpha\beta = \dfrac{q}{1}$

$= -p$ $\qquad = q$

Now $\quad \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$\begin{aligned} &= (-p)^2 - 2(q) \\ &= p^2 - 2q \end{aligned}$$

(ii) $\alpha^3\beta+\alpha\beta^3$

Solution:

$\alpha^3\beta+\alpha\beta^3$

$x^2+px+q=0$

Here a=1, b=p, c=q

Let α, β be the roots of the given equation

Then $\alpha+\beta = -\dfrac{b}{a}$ *and* $\alpha\beta = \dfrac{c}{a}$

$= -\dfrac{p}{1}$ $\qquad \alpha\beta = \dfrac{q}{1}$

$= -p$ $\qquad = q$

Now $\quad \alpha^3\beta+\alpha\beta^3 = \alpha\beta(\alpha^2+\beta^2)$

$$\begin{aligned} &= \alpha\beta\left[(\alpha+\beta)^2 - 2\alpha\beta\right] \\ &= q\left[(-p)^2 - 2q\right] \end{aligned}$$

$$=q\left(p^2-2q\right)$$

(iii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

Solution:

$\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

$x^2+px+q=0$

Here a=1, b=p, c=q

Let α,β be the roots of the given equation

$$\text{Then}\quad \alpha+\beta=-\frac{b}{a}\quad \textit{and}\quad \alpha\beta=\frac{c}{a}$$
$$=-\frac{p}{1}\qquad \alpha\beta=\frac{q}{1}$$
$$=-p\qquad =q$$

$$\text{Now}\quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$$
$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$
$$=\frac{(-p)^2-2q}{q}=\frac{1}{q}\left(p^2-2q\right)$$

2. If α, β are the roots of the equation 4x²-5x+6=0, then find the values of

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

Solution:

$\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α,β be the roots of the given equation

$$\text{Then}\quad \alpha+\beta=-\frac{b}{a}\quad \textit{and}\quad \alpha\beta=\frac{c}{a}$$
$$=-\frac{(-5)}{4}\qquad \alpha\beta=\frac{6}{4}$$
$$=\frac{5}{4}\qquad =\frac{3}{2}$$

$$\text{Now}\quad \frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$
$$=\frac{\frac{5}{4}}{\frac{3}{2}}=\frac{5}{4}\times\frac{2}{3}=\frac{5}{6}$$

(ii) $\alpha^2\beta^2$

Solution:

$\alpha^2\beta^2$

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α,β be the roots of the given equation

$$\begin{array}{lll} \text{Then} \quad \alpha+\beta=-\frac{b}{a} & \textit{and} & \alpha\beta=\frac{c}{a} \\ =-\frac{(-5)}{4} & & \alpha\beta=\frac{6}{4} \\ =\frac{5}{4} & & =\frac{3}{2} \end{array}$$

Now $\quad \alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$

(iii) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$

Solution:

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α,β be the roots of the given equation

$$\begin{array}{lll} \text{Then} \quad \alpha+\beta=-\frac{b}{a} & \textit{and} & \alpha\beta=\frac{c}{a} \\ =-\frac{(-5)}{4} & & \alpha\beta=\frac{6}{4} \\ =\frac{5}{4} & & =\frac{3}{2} \end{array}$$

Now $\quad \frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}=\frac{\alpha+\beta}{\alpha^2\beta^2}=\frac{\alpha+\beta}{(\alpha\beta)^2}$

$$=\frac{\frac{5}{4}}{\left(\frac{3}{2}\right)^2}=\frac{\frac{5}{4}}{\frac{9}{4}}=\frac{5}{4}\times\frac{4}{9}=\frac{5}{9}$$

(iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Solution:

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α,β be the roots of the given equation

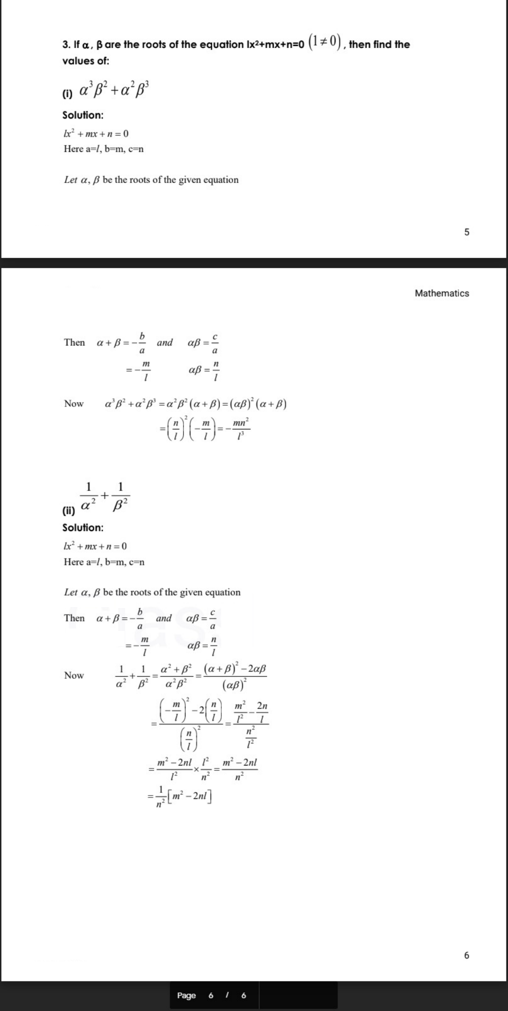
$$\begin{array}{lll} \text{Then} \quad \alpha+\beta=-\frac{b}{a} & \textit{and} & \alpha\beta=\frac{c}{a} \\ =-\frac{(-5)}{4} & & \alpha\beta=\frac{6}{4} \\ =\frac{5}{4} & & =\frac{3}{2} \end{array}$$

Now $\quad \frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$=\frac{\left(\frac{5}{4}\right)^3-3\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}{\frac{3}{2}}=\frac{\frac{125}{64}-\frac{45}{8}}{\frac{3}{2}}$$

$$=\frac{125-360}{64}\times\frac{2}{3}=-\frac{235}{64}\times\frac{2}{3}$$

$$=-\frac{235}{96}$$



3. If α, β are the roots of the equation $lx^2+mx+n=0$ $(1 \neq 0)$, then find the values of:

(i) $\alpha^3\beta^2+\alpha^2\beta^3$

Solution:

$lx^2+mx+n=0$

Here a=*l*, b=m, c=n

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{m}{l}$ $\alpha\beta=\frac{n}{l}$

Now $\alpha^3\beta^2+\alpha^2\beta^3=\alpha^2\beta^2(\alpha+\beta)=(\alpha\beta)^2(\alpha+\beta)$

$$=\left(\frac{n}{l}\right)^2\left(-\frac{m}{l}\right)=-\frac{mn^2}{l^3}$$

(ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solution:

$lx^2+mx+n=0$

Here a=*l*, b=m, c=n

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{m}{l}$ $\alpha\beta=\frac{n}{l}$

Now $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$$=\frac{\left(-\frac{m}{l}\right)^2-2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2}=\frac{\frac{m^2}{l^2}-\frac{2n}{l}}{\frac{n^2}{l^2}}$$

$$=\frac{m^2-2nl}{l^2}\times\frac{l^2}{n^2}=\frac{m^2-2nl}{n^2}$$

$$=\frac{1}{n^2}\left[m^2-2nl\right]$$

Formation of a quadratic equation:

If α and β are the roots of the required quadratic equation.

Let $x=\alpha$ and $x=\beta$

i.e., $x-\alpha=0$, $x-\beta=0$

and $(x-\alpha)(x-\beta)=0$

$x^2-(\alpha+\beta)x+\alpha\beta=0$

which is the required quadratic equation in standard form.

Find a quadratic equation from given roots and establish the formula

$x^2-(\text{sum of the roots})x+\text{product of the roots}=0$

Let α,β be the roots of the given equation

$ax^2+bx+c=0$, $(a\neq 0)$ ....(i)

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

Rewrite eq.(i) as $x^2+\frac{b}{a}x+\frac{c}{a}=0$

or $x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0$

$x^2-(\alpha+\beta)x+\alpha\beta=0$

or $x^2-(\text{sum of the roots})x+\text{product of the roots}=0$, that is

$x^2-Sx+P=0$ where $S=\alpha+\beta$ and $P=\alpha\beta$

Exercise 2.5

1. Write the quadratic equations having following roots.

(a) 1,5

Solution:

Since 1 and 5 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 1 + 5 = 6

P = Product of roots = (1) (5) = 5

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 6x + 5 = 0$

(b) 4,9

Solution:

Since 4 and 9 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 4 + 9 = 13

P = Product of roots = (4) (9) = 36

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 13x + 36 = 0$

(c) -2,3

Solution:

Since -2 and 3 are the roots of the required quadratic equation, therefore,

S = Sum of roots = $-2 + 3 = 1$

P = Product of roots = $(-2)(3) = -6$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (1)x + (-6) = 0$

$x^2 - x - 6 = 0$

(d) 0, – 3

Solution:

Since 0 and – 3 are the roots of the required quadratic equation, therefore,

S = Sum of roots = $0 + (-3) = -3$

P = Product of roots = $(0)(-3) = 0$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (-3)x + 0 = 0$

$x^2 + 3x = 0$

(e) 2, – 6

Solution:

Since 2 and – 6 are the roots of the required quadratic equation, therefore,

S = Sum of roots = $2 + (-6) = -4$

P = Product of roots = $(2)(-6) = -12$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (-4)x + (-12) = 0$

$x^2 + 4x - 12 = 0$

(f) – 1, – 7

Solution:

Since –1 and – 7 are the roots of the required quadratic equation, therefore,

S = Sum of roots = $(-1) + (-7) = -1 - 7 = -8$

P = Product of roots = $(-1)(-7) = 7$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (-8)x + 7 = 0$

$x^2 + 8x + 7 = 0$

(g) 1+ i, 1 – i

Solution:

Since 1+ i and 1 – i are the roots of the required quadratic equation, therefore,

S = Sum of roots = $(1+ i) + (1 - i) = 2$

P = Product of roots = $(1+ i)(1 - i) = 1 - i^2 = 1 - (-1) = 2$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 2x + 2 = 0$

(h) 3+ $\sqrt{2}$, 3 – $\sqrt{2}$

Solution:

Since 3+ $\sqrt{2}$ and 3 – $\sqrt{2}$ are the roots of the required quadratic equation, therefore,

S = Sum of roots = $3 + \sqrt{2} + 3 - \sqrt{2} = 6$

P = Product of roots = $(3 + \sqrt{2})(3 - \sqrt{2}) = (3)^2 - (\sqrt{2})^2 = 9 - 2 = 7$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 6x + 7 = 0$

2. If α, β are the roots of the equation $x^2 - 3x + 6 = 0$

Form equations whose roots are

a) $2\alpha + 1, 2\beta + 1$

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b=–3, c=6

Let α, β be the roots of the given equation

$$\begin{array}{lll} \text{Then} \quad \alpha + \beta = -\dfrac{b}{a} & \textit{and} & \alpha\beta = \dfrac{c}{a} \\ \qquad\qquad = -\dfrac{(-3)}{1} & & \alpha\beta = \dfrac{6}{1} \\ \qquad\qquad = 3 & & = 6 \end{array}$$

Let roots of the new equation be $2\alpha + 1,\ 2\beta + 1$

$$\begin{array}{ll} S = \text{Sum of roots} & P = \text{Product of roots} \\ = (2\alpha + 1) + (2\beta + 1) & = (2\alpha + 1)(2\beta + 1) \\ = 2\alpha + 1 + 2\beta + 1 & = (2\alpha + 1)(2\beta + 1) \\ = 2\alpha + 2\beta + 2 & = 4\alpha\beta + 2\alpha + 2\beta + 1 \\ = 2(\alpha + \beta) + 2 & = 4(6) + 2(3) + 1 \\ = 2(3) + 2 & = 24 + 6 + 1 \\ = 8 & = 31 \end{array}$$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 8x + 31 = 0$

b) $\alpha^2,\ \beta^2$

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b= − 3, c=6

Let α, β be the roots of the given equation

$$\text{Then}\quad \begin{aligned} \alpha+\beta &= -\frac{b}{a} \\ &= -\frac{(-3)}{1} \\ &= 3 \end{aligned} \quad \textit{and} \quad \begin{aligned} \alpha\beta &= \frac{c}{a} \\ \alpha\beta &= \frac{6}{1} \\ &= 6 \end{aligned}$$

Let roots of the new equation be α^2, β^2

$$\begin{aligned} S &= \text{Sum of roots} \\ &= \alpha^2+\beta^2 \\ &= (\alpha+\beta)^2 - 2\alpha\beta \\ &= (3)^2 - 2(6) \\ &= 9-12 \\ &= -3 \end{aligned} \qquad \begin{aligned} P &= \text{Product of roots} \\ &= (\alpha^2)(\beta^2) \\ &= (\alpha^2\beta^2) \\ &= (\alpha\beta)^2 \\ &= (6)^2 \\ &= 36 \end{aligned}$$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (-3)x + 36 = 0$

$x^2 + 3x + 36 = 0$

c) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b= − 3, c=6

Let α, β be the roots of the given equation

$$\text{Then}\quad \begin{aligned} \alpha+\beta &= -\frac{b}{a} \\ &= -\frac{(-3)}{1} \\ &= 3 \end{aligned} \quad \textit{and} \quad \begin{aligned} \alpha\beta &= \frac{c}{a} \\ \alpha\beta &= \frac{6}{1} \\ &= 6 \end{aligned}$$

Let roots of the new equation be $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

$$\begin{aligned} S &= \text{Sum of roots} \\ &= \frac{1}{\alpha}+\frac{1}{\beta} \\ &= \frac{\alpha+\beta}{\alpha\beta} \\ &= \frac{3}{6} = \frac{1}{2} \end{aligned} \qquad \begin{aligned} P &= \text{Product of roots} \\ &= \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) \\ &= \frac{1}{\alpha\beta} \\ &= \frac{1}{6} \end{aligned}$$

Thus, the required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \frac{1}{2}x + \frac{1}{6} = 0$$

$$\Rightarrow 6x^2 - 3x + 1 = 0$$

d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b=−3, c=6

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$= -\frac{(-3)}{1} \qquad \alpha\beta = \frac{6}{1}$$

$$= 3 \qquad = 6$$

Let roots of the new equation be $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

S = Sum of roots $\qquad$ P = Product of roots

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \qquad = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)$$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta} \qquad = 1$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(3)^2 - 2(6)}{6}$$

$$= \frac{9 - 12}{6}$$

$$= -\frac{3}{6}$$

$$= -\frac{1}{2}$$

Thus, the required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \left(-\frac{1}{2}\right)x + 1 = 0$$

$$\Rightarrow 2x^2 + x + 2 = 0$$

e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b=−3, c=6

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$= -\frac{(-3)}{1} \qquad \alpha\beta = \frac{6}{1}$$

$$= 3 \qquad = 6$$

Let roots of the new equation be $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

S = Sum of roots and P = Product of roots

$$= \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} \qquad = (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$= (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta} \qquad = (\alpha + \beta)\left(\frac{\alpha + \beta}{\alpha\beta}\right)$$

$$=3+\frac{3}{6} \qquad\qquad =(3)\left(\frac{3}{6}\right)$$

$$=3+\frac{1}{2} \qquad\qquad =\frac{3}{2}$$

$$=\frac{7}{2}$$

Thus, the required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

$$\Rightarrow 2x^2 - 7x + 3 = 0$$

3. If α, β are the roots of the equation $x^2 + px + q = 0$

Form equations whose roots are

a) α^2, β^2

Solution:

$$x^2 + px + q = 0$$

Here $a = 1,\ b = p,\ c = q$

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$= -\frac{p}{1} \qquad \alpha\beta = \frac{q}{1}$$

$$= -p \qquad = q$$

Let roots of the new equation be α^2, β^2

$S =$ Sum of roots and $P =$ Product of roots

$$= \alpha^2 + \beta^2 \qquad = (\alpha^2)(\beta^2)$$

$$= (\alpha + \beta)^2 - 2\alpha\beta \qquad = (\alpha\beta)^2$$

$$= (-p)^2 - 2q \qquad = q^2$$

$$= p^2 - 2q$$

Thus, the required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - (p^2 - 2q)x + q^2 = 0$$

b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution:

$$x^2 + px + q = 0$$

Here $a = 1,\ b = p,\ c = q$

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$= -\frac{p}{1} \qquad \alpha\beta = \frac{q}{1}$$

$$= -p \qquad = q$$

Let roots of the new equation be $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

$S =$ Sum of roots $\qquad P =$ Product of roots

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \qquad = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)$$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta} \qquad = 1$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-p)^2 - 2q}{q}$$

$$= \frac{p^2 - 2q}{q}$$

Thus, the required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \left(\frac{p^2 - 2q}{q}\right)x + 1 = 0$$

$$qx^2 - \left(p^2 - 2q\right)x + q = 0$$

Synthetic Division:

Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial. In fact, synthetic division is simply a shortcut of long division method.

Exercise 2.6

1. Use synthetic division to find the quotient and the remainder, when

(i) $\left(x^2-7x-1\right)\div\left(x+1\right)$

Solution:

$\left(x^2-7x-1\right)\div\left(x+1\right)$

As $x+1=x-(-1), so\ a=-1$

Now write the coefficients of dividend in a row and a=−1 on the left side.,

$$\begin{array}{r|rrr} & 1 & 7 & -1 \\ -1 & \downarrow & -1 & -6 \\ \hline & 1 & 6 & |-7 \end{array}$$

Quotient $=Q(x)=x+6$

Remainder $=R=-7$

(ii) $\left(4x^3-5x+15\right)\div\left(x+3\right)$

Solution:

As $x+3=x-(-3), so\ a=-3$

Now write the coefficients of dividend in a row and a=−3 on the left side.

$$\begin{array}{r|rrrr} & 4 & 0 & -5 & 15 \\ -3 & \downarrow & -12 & 36 & -93 \\ \hline & 4 & -12 & 31 & |-78 \end{array}$$

Quotient $=Q(x)=4x^2-12x+31$

Remainder $=R=-78$

(iii) $\left(x^3+x^2-3x+2\right)\div\left(x-2\right)$

Solution:

$\left(x^3+x^2-3x+2\right)\div\left(x-2\right)$

As $x-2,\ so\ a=2$

Now write the coefficients of dividend in a row and $a=2$ on the left side.

$$\begin{array}{r|rrrr} & 1 & 1 & -3 & 2 \\ 2 & \downarrow & 2 & 6 & 6 \\ \hline & 1 & 3 & 3 & |8 \end{array}$$

Quotient $=Q(x)=x^2+3x+3$

Remainder $=R=8$

2. Find the value of h using synthetic division, if

(i) **3 is the zero of the polynomial** $2x^3 - 3hx^2 + 9$

Solution:

$P(x) = 2x^3 - 3hx^2 + 9$ and its zero is 3.

Then by synthetic division.

$$\begin{array}{c|cccc} & 2 & -3h & 0 & 9 \\ 3 & \downarrow & 6 & 18-9h & 54-27h \\ \hline & 2 & 6-3h & 18-9h & |\,63-27h \end{array}$$

Remainder $= 63 - 27h$
Since 3 is the zero of the polynomial, therefore
Remainder $= 0$, that is

$$63 - 27h = 0$$
$$27h = 63$$
$$h = \frac{63}{27} = \frac{7}{3}$$

(ii) **1 is the zero of the polynomial** $x^3 - 2hx^2 + 11$

Solution:

$P(x) = x^3 - 2hx^2 + 11$ and its zero is 1.

Then by synthetic division.

$$\begin{array}{c|cccc} & 1 & -2h & 0 & 11 \\ 1 & \downarrow & 1 & 1-2h & 1-2h \\ \hline & 1 & 1-2h & 1-2h & |\,12-2h \end{array}$$

Remainder $= 12 - 2h$
Since 1 is the zero of the polynomial, therefore
Remainder $= 0$, that is

$$12 - 2h = 0$$
$$2h = 12$$
$$h = \frac{12}{2} = 6$$

(iii) **-1 is the zero of the polynomial** $2x^3 + 5hx - 23$.

Solution:

$P(x) = 2x^3 + 5hx - 23$ and its zero is -1.

Then by synthetic division.

$$\begin{array}{c|cccc} & 2 & 0 & 5h & -23 \\ -1 & \downarrow & -2 & 2 & -5h-2 \\ \hline & 2 & -2 & 5h+2 & |\,-5h-25 \end{array}$$

Remainder $= -5h-25$

Since -1 is the zero of the polynomial, therefore

Remainder $= 0$, that is

$$-5h-25=0$$
$$-5h=25$$
$$h=\frac{25}{-5}=-5$$

3. Use synthetic division to find the values of l and m, if

(i) (x+3) and (x-2) are the factors of the polynomial

$$x^3-4x^2+2lx+m$$

Solution:

Since $(x+3)$ and $(x-2)$ are the factors of $\mathrm{p}(x)=x^3+4x^2+2lx+m$.

Therefore -3 and 2 are zeroes of polynomial $\mathrm{p}(x)$.

Now by synthetic division,

$$\begin{array}{r|cccc} & 1 & 4 & 2l & m \\ -3 & \downarrow & -3 & -3 & -6l+9 \\ \hline & 1 & 1 & 2l-3 & |\,m-6l+9 \end{array}$$

Remainder $= m-6l+9$

Since -3 is the zero of the polynomial, therefore,

Remainder $= 0$, that is,

$m-6l+9=0 \qquad ....(i)$

and

$$\begin{array}{r|cccc} & 1 & 4 & 2l & m \\ 2 & \downarrow & 2 & 12 & 4l+24 \\ \hline & 1 & 6 & 2l+12 & |\,m+4l+24 \end{array}$$

Remainder $= m+4l+24$

Since 2 is the zero of polynomial, therefore,

Remainder $= 0$, that is ,

$m+4l+24=0 \qquad ....(ii)$

Now, subtact eq(ii) from eq(i), we get

$$\begin{array}{r} m-6l+9=0 \\ \underline{\pm m \pm 4l \pm 24=0} \\ -10l-15=0 \\ -10l=15 \\ l=-\frac{15}{10} \\ l=-\frac{3}{2} \end{array}$$

Put $l=-\frac{3}{2}$ in eq.(i), we get

$$m-6\left(-\frac{3}{2}\right)+9=0$$
$$m+9+9=0$$
$$m+18=0$$
$$m=-18$$

Thus, $l=-\frac{3}{2},\ m=-18$

(ii) (x-1) and (x+1) are the factors of the polynomial

$$x^3-3lx^2+2mx+6.$$

Solution:

Since $(x-1)$ and $(x+1)$ are the factors of $\mathrm{p}(x)=x^3-3lx^2+2mx+6$.

Therefor 1 and -1 are zeroes of polynomial $p(x)$.

Now by synthetic division,

$$\begin{array}{r|cccc} & 1 & -3l & 2m & 6 \\ 1 & \downarrow & 1 & 1-3l & 2m-3l+1 \\ \hline & 1 & 1-3l & 2m-3l+1 & |2m-3l+7 \end{array}$$

Remainder $= 2m-3l+7$
Since 1 is the zero of the polynomial, therefore,
Remainder $= 0$, that is,
$2m-3l+7=0 \quad ....(i)$
and

$$\begin{array}{r|cccc} & 1 & -3l & 2m & 6 \\ -1 & \downarrow & -1 & 3l+1 & -2m-3l-1 \\ \hline & 1 & -3l-1 & 2m+3l+1 & |-2m-3l+5 \end{array}$$

Remainder $= -2m-3l+5$
Since -1 is the zero of polynomial, therefore,
Remainder $= 0$, that is ,

$-2m-3l+5=0 \qquad ....(ii)$
Add eq(ii) in eq(i), we get

$$\begin{array}{r} 2m-3l+7=0 \\ \underline{-2m-3l+5=0} \\ -6l+12=0 \\ -6l=-12 \\ l=\dfrac{12}{6} \\ l=2 \end{array}$$

Put $l=2$ in eq.(i), we get

$$\begin{aligned} &2m-3(2)+7=0 \\ &2m-6+7=0 \\ &2m+1=0 \\ &2m=-1 \\ &m=-\frac{1}{2} \end{aligned}$$

Thus, $l=2, m=-\frac{1}{2}$

4. Solve by using synthetic division, if

(i) 2 is the root of the equation $x^3-28x+48=0$

Solution:

Since 2 is the root of the equation $x^3-28x+48=0$

Then by synthetic division, we get

$$\begin{array}{r|rrrr} & 1 & 0 & -28 & 48 \\ 2 & \downarrow & 2 & 4 & -48 \\ \hline & 1 & 2 & -24 & |0 \end{array}$$

The depressed equation is

$x^2+2x-24=0$

$x^2+6x-4x-24=0$

$x(x+6)-4(x+6)=0$

$(x-4)(x+6)=0$

Either $x-4=0$ *or* $x+6=0$

$x=4$ $x=-6$

Hence, $-6, 2$ and 4 are the roots of the given equation.

(ii) 3 is the root of the equation $2x^3-3x^2-11x+6=0$.

Solution:

Since 3 is the root of the equation $2x^3-3x^2-11x+6=0$

Then by synthetic division, we get

$$\begin{array}{r|rrrr} & 2 & -3 & -11 & 6 \\ 3 & \downarrow & 6 & 9 & -6 \\ \hline & 2 & 3 & -2 & |0 \end{array}$$

The depressed equation is

$2x^2+3x-2=0$

$2x^2+4x-x-2=0$

$2x(x+2)-1(x+2)=0$

$(2x-1)(x+2)=0$

Either $2x-1=0$ *or* $x+2=0$

$2x=1$ $x=-2$

$x=\frac{1}{2}$

Hence, $-2, \frac{1}{2}$ and 3 are the roots of the given equation.

(iii) -1 is the root of the equation $4x^3-x^2-11x-6=0$.

Solution:

Since -1 is the root of the equation $4x^3-x^2-11x-6=0$

Then by synthetic division, we get

$$\begin{array}{r|rrrr} & 4 & -1 & -11 & -6 \\ -1 & \downarrow & -4 & 5 & 6 \\ \hline & 4 & -5 & -6 & |0 \end{array}$$

The depressed equation is

$4x^2-5x-6=0$

$4x^2-8x+3x-6=0$

$4x(x-2)+3(x+2)=0$

$(4x+3)(x-2)=0$

Either $4x+3=0$ *or* $x-2=0$

$4x=-3$ $x=2$

$x=-\frac{3}{4}$

Hence, $-\frac{3}{4}, -1$ and 2 are the roots of the given equation.

5. Solve by using synthetic division, if

(i) 1 and 3 the roots of the equation $x^4 - 10x^2 + 9 = 0$.

Solution:

Since 1 and 3 are the root of the equation

$x^4 - 10x^2 + 9 = 0$

Then by synthetic division, we get

$$\begin{array}{r|rrrrr}
 & 1 & 0 & -10 & 0 & 9 \\
1 & \downarrow & 1 & 1 & -9 & -9 \\
\hline
 & 1 & 1 & -9 & -9 & \vert\, 0 \\
3 & \downarrow & 3 & 12 & 9 & \\
\hline
 & 1 & 4 & 3 & \vert\, 0 &
\end{array}$$

The depressed equation is

$x^2 + 4x + 3 = 0$

$x^2 + 3x + x + 3 = 0$

$x(x+3) + 1(x+3) = 0$

$(x+1)(x+3) = 0$

Either $x + 1 = 0$ *or* $x + 3 = 0$

$x = -1$ $\quad$ $x = -3$

Thus, $-3, -1, 1$ and 3 are the roots of the given equation.

(ii) 3 and -4 are the roots of the equation $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$.

Solution:

Since 3 and -4 are the root of the equation

$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$

Then by synthetic division, we get

$$\begin{array}{r|rrrrr}
 & 1 & 2 & -13 & -14 & 24 \\
3 & \downarrow & 3 & 15 & 6 & -24
\end{array}$$

	1	5	2	-8	0
-4	↓	-4	-4	8	
	1	1	-2	0	

The depressed equation is

$x^2+x-2=0$

$x^2+2x-x-2=0$

$x(x+2)-1(x+2)=0$

$(x-1)(x+2)=0$

Either $x-1=0$ *or* $x+2=0$

$x=1$ $x=-2$

Thus, $-4,-2,1$ and 3 are the roots of the given equation.

Simultaneous equations:

A system of equations having a common solution is called a system of simultaneous equations.

The set of all the ordered pairs (x,y), which satisfies the system of equations is called the solution set of the system.

Exercise 2.7

Solve the following simultaneous equations.

1. $x + y = 5$; $x^2 - 2y - 14 = 0$

Solution:

$x + y = 5$ ....(i)

$x^2 - 2y - 14 = 0$ ....(ii)

From eq. (i), we have

$y = 5 - x$ ....(iii)

Put value of y in eq.(ii), we get

$$x^2 - 2(5 - x) - 14 = 0$$
$$x^2 - 10 + 2x - 14 = 0$$
$$x^2 + 2x - 24 = 0$$
$$x^2 + 6x - 4x - 24 = 0$$
$$x(x+6) - 4(x+6) = 0$$
$$(x-4)(x+6) = 0$$

Either $x - 4 = 0$ *or* $x + 6 = 0$

$x = 4$ $\quad$ $x = -6$

Put $x = 4$ in eq.(iii), we get Put $x = -6$ in eq.(iii), we get

$y = 5 - 4$ $\quad$ $y = 5 - (-6)$

$= 1$ $\quad$ $= 5 + 6$

$= 11$

The ordered pairs are $(4,1)$ and $(-6,11)$.

Thus, solution set $= \{(4,1), (-6,11)\}$

2. $3x - 2y = 1$; $x^2 + xy - y^2 = 1$

Solution:

$3x - 2y = 1$ ....(i)

$x^2 + xy - y^2 = 1$ ....(ii)

From *eq.* (i), we have

$$2y = 3x - 1$$
$$y = \frac{1}{2}(3x - 1)$$
$$y = \frac{3}{2}x - \frac{1}{2} \quad \text{........}(iii)$$

Put value of y in eq.(ii), we get

$x^2+x\left(\frac{3}{2}x-\frac{1}{2}\right)-\left(\frac{3}{2}x-\frac{1}{2}\right)^2=1$

$x^2+\frac{3}{2}x^2-\frac{1}{2}x-\frac{9}{4}x^2+\frac{3}{2}x-\frac{1}{4}=1$

Multiplying both sides by '4' we get

$4x^2+6x^2-2x-9x^2+6x-1=4$

$4x^2+6x^2-9x^2-2x+6x-1-4=0$

$x^2+4x-5=0$

$x^2+5x-x-5=0$

$x(x+5)-1(x+5)=0$

$(x-1)(x+5)=0$

Either $x-1=0$ *or* $x+5=0$

$x=1$ $\qquad$ $x=-5$

Put $x=1$ in eq.(iii), we get

$$\begin{aligned} y&=\frac{3}{2}(1)-\frac{1}{2}\\ &=\frac{3}{2}-\frac{1}{2}\\ &=\frac{2}{2}\\ &=1 \end{aligned}$$

Put $x=-5$ in eq.(iii), we get

$$\begin{aligned} y&=\frac{3}{2}(-5)-\frac{1}{2}\\ &=-\frac{15}{2}-\frac{1}{2}\\ &=-\frac{16}{2}\\ &=-8 \end{aligned}$$

The ordered pairs are $(1,1)$ and $(-5,-8)$

Thus, solution set $=\{(1,1),(-5,-8)\}$

3. $x-y=7$; $\frac{2}{x}-\frac{5}{y}=2$

Solution:

$x-y=7$ ....(i)

$\frac{2}{x}-\frac{5}{y}=2$

$\frac{2y-5x}{xy}=2$

$2y-5x=2xy$ ....(ii)

From *eq.* (i), we have

$y=x-7$ ....(iii)

Put value of y in eq.(ii), we get

$2(x-7)-5x=2x(x-7)$

$2x-14-5x=2x^2-14x$

$-3x-14=2x^2-14x$

$2x^2-14x+3x+14=0$

$$2x^2-11x+14=0$$
$$2x^2-7x-4x+14=0$$
$$x(2x-7)-2(2x-7)=0$$
$$(x-2)(2x-7)=0$$

Either $x-2=0$ *or* $2x-7=0$

$x=2$ $\qquad 2x=7$

$x=\frac{7}{2}$

Put $x=2$ in eq.(iii), we get Put $x=\frac{7}{2}$ in eq.(iii), we get

$y=2-7$ $\qquad y=\frac{7}{2}-7$

$=-5$ $\qquad =-\frac{7}{2}$

The ordered pairs are $(2,-5)$ and $\left(\frac{7}{2},-\frac{7}{2}\right)$

Thus, solution set $=\left\{(2,-5),\left(\frac{7}{2},-\frac{7}{2}\right)\right\}$

4. $x+y=a-b$; $\frac{a}{x}-\frac{b}{y}=2$

Solution:

$x+y=a-b$ ....(i)

$\frac{a}{x}-\frac{b}{y}=2$

$\frac{ay-bx}{xy}=2$

$ay-bx=2xy$........(ii)

From $eq.(i)$, we have

$y=a-b-x$ ....(iii)

Put value of y in eq.(ii), we get

$$a(a-b-x)-bx=2x(a-b-x)$$
$$a(a-b-x)-bx-2x(a-b-x)=0$$
$$a^2-ab-ax-bx-2ax+2bx+2x^2=0$$
$$2x^2-2ax-ax+2bx-bx+a^2-ab=0$$
$$2x^2-3ax+bx+a^2-ab=0$$
$$2x^2-(3a-b)x+\left(a^2-ab\right)=0$$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-[-(3a-b)]\pm\sqrt{[-(3a-b)]^2-4(2)\left(a^2-ab\right)}}{2(2)}$$

$$=\frac{(3a-b)\pm\sqrt{9a^2-6ab+b^2-8a^2+8ab}}{4}$$

$$=\frac{(3a-b)\pm\sqrt{a^2+2ab+b^2}}{4}$$

$$=\frac{(3a-b)\pm\sqrt{(a+b)^2}}{4}=\frac{(3a-b)\pm(a+b)}{4}$$

Either $x=\frac{(3a-b)+(a+b)}{4}$ *or* $\frac{(3a-b)-(a+b)}{4}$

$x=\frac{3a-b+a+b}{4}$ *or* $\frac{3a-b-a-b}{4}$

$x=\frac{4a}{4}$ *or* $\frac{2a-2b}{4}$

$x = a$ *or* $\frac{2(a-b)}{4} = \frac{a-b}{2}$

Put $x = a$ in eq.(iii), we get

$y = a - b - a$

$= -b$

Put $x = \frac{a-b}{2}$ in eq.(iii), we get

$y = (a-b) - \left(\frac{a-b}{2}\right)$

$= \frac{2a - 2b - a + b}{2}$

$= \frac{a-b}{2}$

$\therefore$ The ordered pairs are $(a, -b)$ and $\left(\frac{a-b}{2}, \frac{a-b}{2}\right)$.

Thus, solution set $= \left\{(a,-b), \left(\frac{a-b}{2}, \frac{a-b}{2}\right)\right\}$

5. $x^2 + (y-1)^2 = 10\ ;\ x^2 + y^2 + 4x = 1$

Solution:

$x^2 + (y-1)^2 = 10$

$x^2 + y^2 - 2y + 1 = 10$

$x^2 + y^2 - 2y = 9 \qquad \text{........}(i)$

$x^2 + y^2 + 4x = 1 \qquad \text{.......}(ii)$

Subtract eq.(ii) from eq.(i), we have

$x^2 + y^2 - 2y = 9$

$\underline{\pm x^2 \pm y^2 \pm 4x = \pm 1}$

$-4x - 2y = 8$

$-2(2x + y) = 8$

$2x + y = -4$

$y = -2x - 4 \qquad \text{......}(iii)$

Put value of y in eq.(ii), we have

$x^2 + (-2x-4)^2 + 4x = 1$

$x^2 + \{-(2x+4)\}^2 + 4x = 1$

$x^2 + 4x^2 + 16x + 16 + 4x - 1 = 0$

$5x^2 + 20x + 15 = 0$

$5(x^2 + 4x + 3) = 0$

$\Rightarrow x^2 + 4x + 3 = 0$

$x^2 + 3x + x + 3 = 0$

$x(x+3) + 1(x+3) = 0$

$(x+1)(x+3) = 0$

Either $x + 3 = 0$ *or* $x + 1 = 0$

$x = -3$ *or* $x = -1$

Put $x=-3$ in eq.(iii), we get

$$\begin{aligned} y &= -2(-3)-4 \\ &= 6-4 \\ &= 2 \end{aligned}$$

Put $x=-1$ in eq.(iii), we get

$$\begin{aligned} y &= -2(-1)-4 \\ &= 2-4 \\ &= -2 \end{aligned}$$

$\therefore$ The ordered pairs are $(-3,2),(-1,-2)$.

Thus, solution set $=\{(-3,2),(-1,-2)\}$

6. $(x+1)^2+(y+1)^2=5$; $(x+2)^2+y^2=5$

Solution:

$(x+1)^2+(y+1)^2=5$

$x^2+2x+1+y^2+2y+1=5$

$x^2+y^2+2x+2y=5-1-1$

$x^2+y^2+2x+2y=3$ ....(i)

$(x+2)^2+y^2=5$

$x^2+4x+4+y^2=5$

$x^2+y^2+4x=5-4$

$x^2+y^2+4x=1$ ....(ii)

Subtract eq.(ii) from eq.(i), we have

$$\begin{array}{l} x^2+y^2+2x+2y=3 \\ \underline{\pm x^2 \pm y^2 \pm 4x \quad = \pm 1} \\ -2x+2y=2 \\ 2(-x+y)=2 \\ -x+y=1 \\ y=x+1 \text{ ......}(iii) \end{array}$$

Put value of y in eq.(ii), we have

$x^2+(x+1)^2+4x=1$

$x^2+x^2+2x+1+4x=1$

$2x^2+6x+1-1=0$

$2x^2+6x=0$

$2x(x+3)=0$

Either $2x=0$ *or* $x+3=0$

$x=0$ *or* $x=-3$

Put $x=0$ in eq.(iii), we get

$$\begin{aligned} y &= 0+1 \\ &= 1 \end{aligned}$$

Put $x=-3$ in eq.(iii), we get

$$\begin{aligned} y &= -3+1 \\ &= -2 \end{aligned}$$

$\therefore$ The ordered pairs are $(0,1),(-3,-2)$.

Thus, solution set $=\{(0,1),(-3,-2)\}$

7. $x^2+2y^2=22$; $5x^2+y^2=29$

Solution:

$x^2+2y^2=22$ ....(i)

$5x^2+y^2=29$ ....(ii)

Multiply eq.(ii) by '2' and then Subtract eq.(ii) from eq.(i), we get

$$\begin{array}{l} \quad x^2+2y^2=22 \\ \underline{\pm 10x^2 \pm 2y^2 = \pm 58} \\ -9x^2=-36 \\ \Rightarrow x^2=4 \end{array}$$

$x = \pm 2$

Put value of $x^2 = 4$ in eq.(i), we get

$4 + 2y^2 = 22$

$2y^2 = 22 - 4$

$2y^2 = 18$

$y^2 = 9$

$y = \pm 3$

Thus, solution set $= \{(\pm 2, \pm 3)\}$

8. $4x^2 - 5y^2 = 6$; $3x^2 + y^2 = 14$

Solution:

$4x^2 - 5y^2 = 6$ ....(i)

$3x^2 + y^2 = 14$ ....(ii)

Multiply eq.(ii) by 5 and then add eq.(i) in eq.(ii), we get

$$\begin{array}{r} 4x^2 - 5y^2 = 6 \\ \underline{15x^2 + 5y^2 = 70} \\ 19x^2 = 76 \end{array}$$

$\Rightarrow x^2 = 4$

$x = \pm 2$

Put value of $x^2 = 4$ in eq.(ii), we get

$3(4) + y^2 = 14$

$12 + y^2 = 14$

$y^2 = 14 - 12$

$y^2 = 2$

$y = \pm\sqrt{2}$

Thus, solution set $= \{(\pm 2, \pm\sqrt{2})\}$

9. $7x^2 - 3y^2 = 4$; $2x^2 + 5y^2 = 7$

Solution:

$7x^2 - 3y^2 = 4$ ....(i)

$2x^2 + 5y^2 = 7$ ....(ii)

Multiply eq.(i) by '5' and eq.(ii) by '3' then add eq.(i) and eq.(ii), we get

$$\begin{array}{r} 35x^2 - 15y^2 = 20 \\ \underline{6x^2 + 15y^2 = 21} \\ 41x^2 = 41 \end{array}$$

$\Rightarrow x^2 = 1$

$x = \pm 1$

Put value of $x^2 = 1$ in eq.(ii), we get

$2(1)+5y^2=7$
$2+5y^2=7$
$5y^2=7-2$

$5y^2=5$
$y^2=1$
$y=\pm\sqrt{1}$

Thus, solution set $=\{(\pm1,\pm1)\}$

10. $x^2+2y^2=3$; $x^2+4xy-5y^2=0$

Solution:

$x^2+2y^2=3$ ....(i)
$x^2+4xy-5y^2=0$
$x^2+5xy-xy-5y^2=0$
$x(x+5y)-y(x+5y)=0$
$(x-y)(x+5y)=0$
$x-y=0$ *or* $x+5y=0$
or $y=x$ $x=-5y$

Put $y=x$ in eq.(i), we get	Put $x=-5y$ in eq.(i), we get
$x^2+2x^2=3$	$(-5y)^2+2y^2=3$
$3x^2=3$	$25y^2+2y^2=3$
$\Rightarrow$ $x^2=1$	$27y^2=3$
$x=\pm1$	$y^2=\frac{1}{9}$
	$y=\pm\frac{1}{3}$
Put $x=1$ in $y=x$, we get	Put $y=\frac{1}{3}$ in $x=-5y$, we get
$y=1$	$x=-5\left(\frac{1}{3}\right)$
Put $x=-1$ in $y=x$, we get	$=-\frac{5}{3}$

$y=-1$	Put $y=-\frac{1}{3}$ in $x=-5y$, we get
	$x=-5\left(-\frac{1}{3}\right)$
	$=\frac{5}{3}$
$\therefore$ The ordered pairs are $(1,1),(-1,-1)$.	$\therefore$ The ordered pairs are $\left(\frac{5}{3},-\frac{1}{3}\right),\left(-\frac{5}{3},\frac{1}{3}\right)$

Thus, solution set $=\left\{(1,1),(-1,-1),\ \left(\frac{5}{3},-\frac{1}{3}\right),\left(-\frac{5}{3},\frac{1}{3}\right)\right\}$

11. $3x^2-y^2=26$; $3x^2-5xy-12y^2=0$

Solution:

$3x^2-y^2=26$ ....(i)
$3x^2-5xy-12y^2=0$
$3x^2-9xy+4xy-12y^2=0$
$3x(x-3y)+4y(x-3y)=0$
$(3x+4y)(x-3y)=0$

Either $3x+4y=0$ *or* $x-3y=0$
$3x=-4y$ $x=3y$

$$x = -\frac{4}{3}y$$

Put $x = 3y$ in eq.(i), we get

$$3(3y)^2 - y^2 = 26$$
$$3(9y^2) - y^2 = 26$$
$$27y^2 - y^2 = 26$$
$$26y^2 = 26$$
$$y^2 = 1$$
$$y = \pm 1$$

Put $y = 1$ in $x = 2y$, we get

$$x = 3(1)$$

$$x = 3$$

Put $y = -1$ in $x = 2y$, we get

$$x = 3(-1)$$
$$x = -3$$

$\therefore$ The ordered pairs are $(3,1),(-3,-1)$

Put $x = -\frac{4}{3}y$ in eq.(i), we get

$$3\left(-\frac{4}{3}y\right)^2 - y^2 = 26$$
$$3\left(\frac{16}{9}y^2\right) - y^2 = 26$$
$$\frac{16}{3}y^2 - y^2 = 26$$
$$\frac{16y^2 - 3y^2}{3} = 26$$
$$13y^2 = 26 \times 3$$
$$y^2 = \frac{78}{13}$$
$$y^2 = 6$$
$$y = \pm\sqrt{6}$$

Put $y = \sqrt{6}$ in $x = -\frac{4}{3}y$, we get

$$x = -\frac{4}{3}\left(\sqrt{6}\right)$$
$$x = \frac{-4\sqrt{6}}{3}$$

Put $y = -\sqrt{6}$ in $x = -\frac{4}{3}y$, we get

$$x = -\frac{4}{3}\left(-\sqrt{6}\right)$$
$$x = \frac{4\sqrt{6}}{3}$$

$\therefore$ The ordered pairs are $\left(\frac{-4\sqrt{6}}{3}, \sqrt{6}\right), \left(\frac{4\sqrt{6}}{3}, -\sqrt{6}\right)$

Thus, solution set $=\left\{(3,1),(-3,-1),\left(\frac{-4\sqrt{6}}{3},\sqrt{6}\right),\left(\frac{4\sqrt{6}}{3},-\sqrt{6}\right)\right\}$

12. $x^2+xy=5 \quad ; \quad y^2+xy=3$

Solution:

$x^2+xy=5 \quad ....(i)$

$y^2+xy=3 \quad ....(ii)$

Multiply eq.(i) by '3' and eq.(ii) by '5' then subtract eq.(ii) *from* eq.(i), we get

$$\begin{array}{l} 3x^2 \quad +3xy=15 \\ \underline{\pm 5y^2 \pm 5xy=\pm 15} \\ 3x^2-2xy-5y^2=0 \end{array}$$

$3x^2-5xy+3xy-5y^2=0$

$x(3x-5y)+y(3x-5y)=0$

$(x+y)(3x-5y)=0$

Either $x+y=0$ *or* $3x-5y=0$

$y=-x \qquad 3x=5y$

$x=\frac{5}{3}y$

Put $y=-x$ in eq.(i), we get

$x^2+x(-x)=5$

$x^2-x^2=5$

$0=5$

Which is not possible

Put $x=\frac{5}{3}y$ in eq.(i), we get

$\left(\frac{5}{3}y\right)^2+\left(\frac{5}{3}y\right)y=5$

$\frac{25}{9}y^2+\frac{5}{3}y^2=5$

$\frac{25y^2+15y^2}{9}=5$

$\frac{40y^2}{9}=5$

$40y^2=45$

$y^2=\frac{45}{40}$

$y^2=\frac{9}{8}$

$y=\pm\frac{3}{2\sqrt{2}}$

Put $y=\frac{3}{2\sqrt{2}}$ in $x=\frac{5}{3}y$, we get

$x=\frac{5}{3}\left(\frac{3}{2\sqrt{2}}\right)$

$x=\frac{5}{2\sqrt{2}}$

Put $y=-\frac{3}{2\sqrt{2}}$ in $x=\frac{5}{3}y$, we get

$x=\frac{5}{3}\left(-\frac{3}{2\sqrt{2}}\right)$

$x=-\frac{5}{2\sqrt{2}}$

$\therefore$ The ordered pairs are $\left(\frac{5}{2\sqrt{2}},\frac{3}{2\sqrt{2}}\right),\left(-\frac{5}{2\sqrt{2}},-\frac{3}{2\sqrt{2}}\right)$

Thus, solution set $=\left\{\left(\frac{5}{2\sqrt{2}},\frac{3}{2\sqrt{2}}\right),\left(-\frac{5}{2\sqrt{2}},-\frac{3}{2\sqrt{2}}\right)\right\}$

13. $x^2-2xy=7 \quad ; \quad xy+3y^2=2$

Solution:

$x^2-2xy=7 \quad ....(i)$

$$xy+3y^2=2 \quad \ldots\ldots..(ii)$$

Multiply eq.(i) by '2' and eq.(ii) by '7' then subtract eq.(ii) *from* eq.(i), we get

$$\begin{array}{l} 2x^2 \qquad -4xy=14 \\ \underline{\pm 21y^2 \pm 7xy = \pm 14} \\ 2x^2-11xy-21y^2=0 \end{array}$$

$$2x^2-14xy+3xy-21y^2=0$$

$$2x(x-7y)+3y(x-7y)=0$$

$$(2x+3y)(x-7y)=0$$

Either $2x+3y=0$ *or* $x-7y=0$

$$2x=-3y \qquad\qquad x=7y$$

$$x=-\frac{3}{2}y$$

Put $x=-\frac{3}{2}y$ in eq.(ii), we get

$$\left(-\frac{3}{2}y\right)y+3y^2=2$$

$$-\frac{3}{2}y^2+3y^2=2$$

$$\frac{-3y^2+6y^2}{2}=2$$

$$3y^2=4$$

$$y^2=\frac{4}{3}$$

$$y=\pm\frac{2}{\sqrt{3}}$$

Put $x=7y$ in eq.(ii), we get

$$(7y)y+3y^2=2$$

$$7y^2+3y^2=2$$

$$10y^2=2$$

$$y^2=\frac{2}{10}$$

$$y=\pm\frac{\sqrt{2}}{\sqrt{10}}$$

$$y=\pm\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{5}}$$

$$y=\pm\frac{1}{\sqrt{5}}$$

Put y $=\frac{2}{\sqrt{3}}$ in $x=-\frac{3}{2}y$, we get

$$x=-\frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)$$

$$x=-\frac{3}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$x=-\frac{3\sqrt{3}}{\left(\sqrt{3}\right)^2}=-\frac{3\sqrt{3}}{3}$$

$$x=-\sqrt{3}$$

Put y $=-\frac{2}{\sqrt{3}}$ in $x=-\frac{3}{2}y$, we get

$$x=-\frac{3}{2}\left(-\frac{2}{\sqrt{3}}\right)$$

$$x=\frac{3}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$x=\frac{3\sqrt{3}}{\left(\sqrt{3}\right)^2}=\frac{3\sqrt{3}}{3}$$

$x = \sqrt{3}$

$\therefore$ The ordered pairs are $\left(-\sqrt{3}, \frac{2}{\sqrt{3}}\right), \left(\sqrt{3}, -\frac{2}{\sqrt{3}}\right)$

Put $y = \frac{1}{\sqrt{5}}$ in $x = 7y$, we get

$x = 7\left(\frac{1}{\sqrt{5}}\right)$

$x = \frac{7}{\sqrt{5}}$

Put $y = -\frac{1}{\sqrt{5}}$ in $x = 7y$, we get

$x = 7\left(-\frac{1}{\sqrt{5}}\right)$

$x = -\frac{7}{\sqrt{5}}$

$\therefore$ The ordered pairs are $\left(\frac{7}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{7}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$

Thus, solution set $= \left\{\left(\frac{7}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{7}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right), \left(-\sqrt{3}, \frac{2}{\sqrt{3}}\right), \left(\sqrt{3}, -\frac{2}{\sqrt{3}}\right)\right\}$

Exercise 2.8

1. **The product of two positive consecutive numbers is 182. Find the numbers.**

Solution:

Let the numbers be $x, x+1$

Then, $(x)(x+1)=182$

$x^2+x-182=0$

$x^2+14x-13x-182=0$

$x(x+14)-13(x+14)=0$

$(x-13)(x+14)=0$

$x-13=0$ *gives* $x=13$

Numbers are: $x=13$

$x+1=13+1=14$

$13,14$

Now, $x+14=0$

gives $x=-14$

we ignore this value because it is negative

2. **The sum of squares of three positive consecutive numbers is 77. Find the numbers.**

Solution:

Let the numbers be $x, x+1, x+2$

Applying the given conditions

$(x)^2+(x+1)^2+(x+2)^2=77$

$x^2+x^2+2x+1+x^2+4x+4=77$

$3x^2+6x+5=77$

$3x^2+6x+5-77=0$

$3x^2+6x-72=0$

$x^2+2x-24=0$ (Dividing by 3)

$x^2+6x-4x-24=0$

$x(x+6)-4(x+6)=0$

$(x-4)(x+6)=0$

$x-4=0$ *gives* $x=4$

Numbers are: $x=4$

$x+1=4+1=5$

$x+2=4+2=6$

$4,5,6$

$x+6=0$

$x=-6$, we ignore the negative number.

3. **The sum of five times a number and the square of the number is 204. Find the number.**

Solution:

Let the numbers be

According to the given conditions,

$$x^2+5x=204$$
$$x^2+5x-204=0$$
$$x^2+17x-12x-204=0$$
$$x(x+17)-12(x+17)=0$$
$$(x+17)(x-12)=0$$
$$x-12=0 \quad \textit{gives} \quad x=12$$
$$\textit{or}\ x+17 \quad \textit{gives} \quad x=-17$$

Number is 12 or -17.

4. The product of five less than three times a certain number and one less than four times the number is 7. Find the number.

Solution:

Let the numbers be x

According to the given conditions,

$$(3x-5)(4x-1)=7$$
$$12x^2-3x-20x+5=7$$
$$12x^2-23x+5=7$$
$$12x^2-23x+5-7=0$$
$$12x^2-23x-2=0$$
$$12x^2-24x+x-2=0$$
$$12x(x-2)+1(x-2)=0$$
$$(12x+1)(x-2)=0$$
$$x-2=0 \quad \textit{gives} \quad x=2$$
$$\textit{and}\ 12x+1=0 \quad \textit{gives} \quad x=-\frac{1}{12}$$

Number is 2 or $-\frac{1}{12}$.

5. The difference of a number and its reciprocal is $\frac{15}{4}$. Find the number.

Solution:

Let the numbers be x

Then, According to the given conditions,

$$x-\frac{1}{x}=\frac{15}{4}$$
$$4x^2-4=15x \quad (\text{Multiplying by } 4\text{x})$$
$$4x^2-15x-4=0$$
$$4x^2-16x+x-4=0$$
$$4x(x-4)+1(x-4)=0$$
$$(x-4)(4x+1)=0$$
$$x-4=0 \qquad gives \quad x=4$$
$$and\ 4x+1=0 \qquad gives \quad x=-\frac{1}{4}$$

Number is 4 and $-\frac{1}{4}$.

6. The sum of squares of two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the number.

Solution:

Let xy be the number, where unit digit is y and tens digit is x.

According the given condition,

$$x^2+y^2=65 \quad \text{..........}(A)$$

$$\text{Number}=y+10x$$
$$y+10x=9(x+y)$$
$$y+10x=9x+9y$$
$$10x-9x=9y-y$$
$$x=8y \qquad \text{........}(B)$$

$$Putting\ x=8y\ in\ (A)$$
$$(8y)^2+y^2=65$$
$$64y^2+y^2=65$$
$$65y^2=65$$
$$y^2=1$$
$$y=\pm1$$
$$y=1 \quad (\text{taking +ve value})$$

$$Putting\ y=1\ in\ (B)$$
$$x=8(1)=8$$

Now,

$$\text{Number}=y+10x$$

$$=1+10(8)$$
$$=81$$

Therefore, number is 81.

7. The sum of the co-ordinates of a point is 9 and sum of their squares is 45. Find the co-ordinates of the point.

Solution:

Let $P(x,y)$ be the point.

According the given condition,

$$x+y=9 \quad \text{..........}(\mathrm{A})$$
and $$x^2+y^2=45 \quad \text{..........}(B)$$
From A $$x=9-y \quad \text{..........}(C)$$

Putting $x=9-y$ in (B)

$$(9-y)^2+y^2=45$$
$$81-18y+y^2+y^2=45$$
$$2y^2-18y+81-45=0$$
$$2y^2-18y+36=0$$
$$y^2-9y+18=0 \qquad (\text{Dividing by } 2)$$
$$y^2-6y-3y+18=0$$
$$y(y-6)-3(y-6)=0$$
$$(y-6)(y-3)=0$$
$y-6=0$ gives $y=6$

Then from (C)
$$x=9-6=3$$
Point is $\mathrm{P}(3,6)$

When y-3=0 then y=3

From (C)
$$x=9-3=6$$

Point is $\mathrm{P}(6,3)$

8. Find two integers whose sum is 9 and the difference of their squares is also 9.

Solution:

Let the integers be x,y.

Then,
$$x+y=9 \quad \text{..........}(\mathrm{A})$$
and $$x^2-y^2=9 \quad \text{..........}(B)$$
From A $x=9-y$

Putting $x=9-y$ in (B)

$$(9-y)^2-y^2=9$$
$$81-18y+y^2-y^2=9$$
$$-18y=9-81$$
$$-18y=-72$$
$$y=-\frac{72}{18}$$
$$y=4$$

Putting $y=4$ in (A)

$$x+4=9$$
$$x=9-4=5$$

Integers are 5,4

9. Find two integers whose difference is 4 and whose squares differ by 72.

Solution:

Let the integers be x,y.

Then,

$$x-y=4 \quad \text{..........(A)}$$

and $$x^2-y^2=72 \quad \text{..........}(B)$$

From A $x-y=4$

$$x=4+y$$

Putting $x=4+y$ in (B), we get

$$(4+y)^2-y^2=72$$
$$16+y^2+8y-y^2=72$$

$$8y=72-16$$
$$8y=56$$
$$y=\frac{56}{8}$$
$$y=7$$

Putting $y=7$ in (A)

$$x-7=4$$
$$x=4+7=11$$

Integers are 11 and 7.

10. Find the dimensions of a rectangle, whose perimeter is 80cm and its area is 375cm².

Solution:

Let x and y be the length and width respectiely of the rectangle.

According to the given conditions,

Perimeter $2x+2y=80$

or $x+y=40 \quad \text{..........(A)}$

Area $xy=375 \quad \text{..........}(B)$

From A $x=40-y$

Putting $x=40-y$ in (B), we get

$$(40-y)y=375$$
$$40y-y^2=375$$
$$\Rightarrow y^2-40y+375=0$$
$$y^2-25y-15y+375=0$$
$$y(y-25)-15(y-25)=0$$
$$(y-15)(y-25)=0$$
$$y-15=0 \quad \textit{gives } y=15$$

Putting $y=15$ in (A)

$$x+15=40$$
$$x=40-15=25$$

Length $=25cm$. Breadth $=15cm$.

Miscellaneous Exercise 2

1. Multiple Choice Questions:

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) If α, β are the roots of $3x^2 + 5x - 2 = 0$, then $\alpha + \beta$ is

(a) $\frac{5}{3}$ **(b)** $\frac{3}{5}$ **(c)** $\frac{-5}{3}$ **(d)** $\frac{-2}{3}$

(ii) If α, β are the roots of $7x^2 - x + 4 = 0$, then $\alpha\ \beta$ is

(a) $\frac{-1}{7}$ **(b)** $\frac{4}{7}$ **(c)** $\frac{7}{4}$ **(d)** $\frac{-4}{7}$

(iii) Roots of equation $4x^2 - 5x + 2 = 0$ are

(a) irrational **(b)** imaginary **(c)** rational **(d)** none

(iv) Cube roots of -1 are

(a) $-1, -\omega, -\omega^2$ **(b)** $-1, \omega, -\omega^2$ **(c)** $-1, -\omega, \omega^2$ **(d)** $1, -\omega, -\omega^2$

(v) Sum of the cube roots of unity is

(a) 0 **(b)** 1 **(c)** -1 **(d)** 3

(vi) Product of cube roots of unity is

(a) 0 **(b)** 1 **(c)** -1 **(d)** 3

(vii) If $b^2 - 4ac < 0$, then the roots of $ax^2 + bx + c = 0$ are

(a) irrational **(b)** rational **(c)** imaginary **(d)** none

(viii) If $b^2 - 4ac > 0$, but not a perfect square then roots of $ax^2 + bx + c = 0$ are

(a) imaginary **(b)** rational **(c)** irrational **(d)** none

(ix) $\frac{1}{\alpha}+\frac{1}{\beta}$ **is equal to**

(a) $\frac{1}{\alpha}$ **(b)** $\frac{1}{\alpha}-\frac{1}{\beta}$ **(c)** $\frac{\alpha-\beta}{\alpha\beta}$ **(d)** $\frac{\alpha+\beta}{\alpha\beta}$

(x) $\alpha^2+\beta^2$ **is equal to**

(a) $\alpha^2-\beta^2$ **(b)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

(c) $(\alpha+\beta)^2-2\alpha\beta$ **(d)** $\alpha+\beta$

(xi) Two square roots of unity are

(a) 1,-1 **(b)** 1, ω **(c)** 1, -ω **(d)** ω, ω^2

(xii) Roots of equation $4x^2 - 4x + 1 = 0$ are

(a) real, equal **(b)** real, unequal **(c)** imaginary **(d)** irrational

(xiii) If α, β are the roots of $px^2 + qx + r = 0$, then sum of roots 2α and 2β is

(a) $\frac{-q}{p}$ **(b)** $\frac{r}{p}$ **(c)** $\frac{-2q}{p}$ **(d)** $\frac{q}{-2p}$

(xiv) If α, β are the roots of $x^2 - x - 1 = 0$, then product of roots 2α and 2β is

(a) -2 **(b)** 2 **(c)** 4 **(d)** -4

(xv) The nature of roots of equation $ax^2 + bx + c = 0$ is determined by

(a) sum of roots **(b)** Product of roots

(c) Synthetic division **(d)** Discriminant

(xvi) The discriminant of $ax^2 + bx + c = 0$ is

(a) $b^2 - 4ac$ **(b)** $b^2 + 4ac$ **(c)** $-b^2 + 4ac$ **(d)** $-b^2 - 4ac$

Answers:

(i)	c	**(ii)**	b	**(iii)**	b	**(iv)**	a	**(v)**	a
(vi)	b	**(vii)**	c	**(viii)**	c	**(ix)**	d	**(x)**	c
(xi)	a	**(xii)**	a	**(xiii)**	c	**(xiv)**	d	**(xv)**	d
(xvi)	a								

Write short answers of the following question.

(i) Discuss the nature of the roots of the following equations.

$(a)\, x^2 + 3x + 5 = 0$ $(b)\, 2x^2 - 7x + 3 = 0$

$(c)\, x^2 + 6x - 1 = 0$ $(d)\, 16x^2 - 8x + 1 = 0$

$(a)\, x^2 + 3x + 5 = 0$

$Here\ a=1, b=3, c=5$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (3)^2-4(1)(5) \\ &= 9-20 \\ &= -11 \end{aligned}$$

Disc. is negative therefore, roots are imaginary.

$(b)\, 2x^2-7x+3=0$

$Here\ a=2, b=-7, c=3$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (-7)^2-4(2)(3) \\ &= 49-24 \\ &= 25=5^2 \end{aligned}$$

Disc. is perfect square, therefore, roots are real, rational and unequal.

$(c)\, x^2+6x-1=0$

$Here\ a=1, b=6, c=-1$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (6)^2-4(1)(-1) \\ &= 36+4 \\ &= 40 \quad (+ve) \end{aligned}$$

Roots are real, irrational and unequal.

$(d)\, 16x^2-8x+1=0$

$Here\ a=16, b=-8, c=1$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (-8)^2-4(16)(1) \\ &= 64-64=0 \end{aligned}$$

Disc. is zero, therefore, roots are real, rational and equal.

(ii) Find ω^2, if $\omega=\dfrac{-1+\sqrt{-3}}{2}$

Solution:

$$\omega=\frac{-1+\sqrt{-3}}{2}$$

$$\omega^2=\left(\frac{-1+\sqrt{-3}}{2}\right)^2$$

$$\omega^2=\frac{-1+\left(\sqrt{-3}\right)^2-2\left(\sqrt{-3}\right)}{4}$$

$$\omega^2=\frac{-1-3-2\sqrt{-3}}{4}$$

$$\omega^2=\frac{-2-2\sqrt{-3}}{4}$$

$$\omega^2 = \frac{2\left(-1-\sqrt{-3}\right)}{4}$$
$$\omega^2 = \frac{-1-\sqrt{-3}}{2}$$

(iii) Prove that the sum of the all cube roots of unity is zero.

Solution:

We know that cube roots of unity are:

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$$

$$\text{Now, } 1+\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$
$$= \frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2}$$
$$= \frac{2-2}{4} = 0 \quad \text{Proved}$$

(iv) Find the product of complex cube roots of unity.

Solution:

Roots are: $1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$

$$\text{Product} = 1\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$$
$$= \frac{(-1)^2-\left(\sqrt{-3}\right)^2}{4}$$
$$= \frac{1-(-3)}{4}$$
$$= \frac{1+3}{4} = \frac{4}{4} = 1$$

(v) Show that $x^3+y^3 = (x+y)(x+\omega y)\left(x+\omega^2 y\right)$.

Solution:

$$
\begin{aligned}
& R.H.S \\
&= (x+y)(x+\omega y)\left(x+\omega^2 y\right) \\
&= (x+y)\left[x^2+\omega^2 xy+\omega xy+\omega^3 y^2\right] \\
&= (x+y)\left[x^2+\left(\omega^2+\omega\right)xy+(1)y^2\right] \\
&= (x+y)\left[x^2+(-1)xy+y^2\right] \\
&= (x+y)\left(x^2-xy+y^2\right) \\
&= x^3-x^2y+xy^2+x^2y-xy^2+y^3 \\
&= x^3+y^3 = L.H.S
\end{aligned}
$$

(vi) Evaluate $\omega^{37}+\omega^{38}+1$.

Solution:

$$\begin{aligned}
&\omega^{37}+\omega^{38}+1\\
&=\left(\omega^{3}\right)^{12}\omega+\left(\omega^{3}\right)^{12}\omega^{2}+1\\
&=(1)^{12}\omega+(1)^{12}\omega^{2}+1\\
&=(1)\omega+(1)\omega^{2}+1\\
&=\omega+\omega^{2}+1\\
&=1+\omega+\omega^{2}=0
\end{aligned}$$

(vii) Evaluate $\left(1-\omega-\omega^{2}\right)^{6}$.

Solution:

$$\begin{aligned}
&\left(1-\omega-\omega^{2}\right)^{6}\\
&=\left[1-\left(\omega+\omega^{2}\right)\right]^{6}\\
&=\left[1-(-1)\right]^{6}\\
&=(1+1)^{6}\\
&=(2)^{6}\\
&=64
\end{aligned}$$

(viii) If ω is cube root of unity, form an equation whose roots are 3ω and 3ω²

Solution:

Roots are $3\omega, 3\omega^{2}$

$S=\text{Sum of roots}=3\omega+3\omega^{2}=3\left(\omega+\omega^{2}\right)$

$P=\text{Product of roots}=3\omega\times3\omega^{2}=9\omega^{3}$

Required equation:

$$x^{2}-Sx+P=0$$

$$\begin{aligned}
&x^{2}-3\left(\omega+\omega^{2}\right)x+9\omega^{3}=0\\
&x^{2}-3(-1)x+9(1)=0\\
&x^{2}+3x+9=0
\end{aligned}$$

(ix) Using synthetic division, find the remainder and quotient when

$$\left(x^{3}+3x^{2}+2\right)\div(x-2)$$

Solution:

$$\begin{aligned}
\textit{Here,}\quad &P(x)=x^{3}+3x^{2}+2\\
&\qquad\;\;=x^{3}+3x^{2}+0x+2\\
&x-a=x-2\\
&\quad a=2
\end{aligned}$$

Using synthetic division:

	1	3	0	2
2	↓	2	10	20
	1	5	10	22

$Quotient = x^2+5x+10$

$\text{Remainder} = 22$

(x) Using synthetic division, show that x-2 is the factor of x^3+x^2-7x+2.

Solution:

$$\begin{aligned} Here,\ & P(x) = x^3+x^2-7x+2 \\ & x-a = x-2 \\ & a = 2 \end{aligned}$$

Using synthetic division:

	1	1	-7	2
2	↓	2	6	-2
	1	3	-1	0

$R = 0$, therefore, $x-2$ is a factor of $P(x)$

(xi) Find the sum and product of the roots of the equation $2px^2+3qx-4r=0$

Solution:

$2px^2+3qx-4r=0$

$$\begin{aligned} Here, a = 2p, b = 3q, c = -4r \\ S = -\frac{b}{a} = -\frac{3q}{2p} \\ P = \frac{c}{a} = \frac{-4r}{2p} = -\frac{2r}{p} \end{aligned}$$

(xii) Find $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ of the roots of the equation $x^2-4x+3=0$.

Solution:

$x^2-4x+3=0$

$$\begin{aligned} Here, S = \alpha+\beta = -\frac{-4}{1} = 4 \\ P = \alpha\beta = \frac{3}{1} = +3 \end{aligned}$$

$$Now,\ \frac{1}{\alpha^2}+\frac{1}{\beta^2}$$

$$=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

Putting values of $\alpha+\beta$ and $\alpha\beta$, we get

$$=\frac{(4)^2-2(3)}{(3)^2}$$

$$=\frac{16-6}{9}=\frac{10}{9}$$

(xiii) If α, β are the roots of 4x² - 3x + 6 = 0, find

$(a)\ \alpha^2+\beta^2$ $\qquad$ $(b)\ \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$(c)\ \alpha-\beta$

Solution:

$Equation: 4x^2-3x+6=0$

$$Here, S=\alpha+\beta=-\left(\frac{-3}{4}\right)=\frac{3}{4}$$

$$P=\alpha\beta=\frac{6}{4}=\frac{3}{2}$$

$(a)\ \alpha^2+\beta^2$

$$=(\alpha+\beta)^2-2$$

Putting values of $\alpha+\beta$ and $\alpha\beta$, we get

$$=\left(\frac{3}{4}\right)^2-2\left(\frac{3}{2}\right)$$

$$=\frac{9}{16}-3$$

$$=\frac{9-48}{16}=\frac{-39}{16}$$

$(b)\ \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

Putting values of $\alpha+\beta$ and $\alpha\beta$, we get

$$=\frac{\left(\frac{3}{4}\right)^2-2\left(\frac{3}{2}\right)}{\frac{3}{2}}$$

$$=\left(\frac{9}{16}-3\right)\div\frac{3}{2}$$

$$=\left(\frac{9-48}{16}\right)\times\frac{2}{3}$$

$$=\frac{-39}{16}\times\frac{2}{3}$$

$$=\frac{-13}{8}$$

$(c)\ \alpha-\beta$

$$\begin{aligned}\alpha-\beta&=\sqrt{(\alpha-\beta)^2}\\&=\sqrt{\alpha^2+\beta^2-2\alpha\beta}\\&=\sqrt{(\alpha+\beta)^2-2\alpha\beta-2\alpha\beta}\\&=\sqrt{(\alpha+\beta)^2-4\alpha\beta}\end{aligned}$$

Putting values of $\alpha+\beta$ and $\alpha\beta$, we get

$$=\sqrt{\left(\frac{3}{4}\right)^2-4\left(\frac{3}{2}\right)}$$

$$\begin{aligned}&=\sqrt{\frac{9}{16}-6}\\&=\sqrt{\frac{9-96}{16}}\\\alpha-\beta&=\sqrt{\frac{-87}{16}}\\\alpha-\beta&=\frac{\sqrt{-87}}{4}\end{aligned}$$

(xiv) If α, β are the roots of $x^2 - 5x + 7 = 0$, find an equation whose roots are

(a) -α, -β **(b) 2α, 2β**

(a) -α, -β

Solution:

$Equation: x^2-5x+7=0$

$Here, a=1, b=-5, c=7$

$S=\text{Sum of roots}=\alpha+\beta=-\frac{b}{a}=-\left(\frac{-5}{1}\right)=5$ ....(A)

$P=\text{Product of roots}=\quad \alpha\beta=\frac{c}{a}=\frac{7}{1}=7$ ....(B)

$$\begin{aligned}\text{Part}(a)\qquad S&=-\alpha-\beta\\&=-(\alpha+\beta)=-(5)=-5 &&\text{from }(A)\\P&=(-\alpha)(-\beta)=\alpha\beta=7 &&\text{from }(B)\end{aligned}$$

Equation:

$$\begin{aligned}&x^2-Sx+P=0\\&x^2-(-5)x+7=0\\&x^2+5x+7=0\end{aligned}$$

(b) 2α, 2β

$$\begin{aligned} S &= 2\alpha + 2\beta \\ &= 2(\alpha+\beta) \\ &= 2(5) = 10 \quad \text{from } (A) \\ P &= (2\alpha)(2\beta) \\ &= 4\alpha\beta = 7 \\ &= 4(7) = 28 \quad \text{from } (B) \end{aligned}$$

Equation:

$$x^2 - Sx + P = 0$$
$$x^2 - 10x + 28 = 0$$

Q3. Fili in the blanks:

(i) The discriminant of $ax^2 + bx + c = 0$ is ________ .

(ii) If $b^2 - 4ac = 0$, then roots of $ax^2 + bx + c = 0$ are ____________ .

(iii) If $b^2 - 4ac > 0$, then the roots of $ax^2 + bx + c = 0$ are ____________.

(iv) If $b^2 - 4ac < 0$, then the roots of $ax^2 + bx + c = 0$ are ____________.

(v) If $b^2 - 4ac > 0$ and perfect square, then the roots of $ax^2 + bx + c = 0$ are_________.

(vi) If $b^2 - 4ac > 0$, and not a perfect square, then roots of $ax^2 + bx + c = 0$ are_______.

(vii) If α, β are the roots of $ax^2 + bx + c = 0$, then sum of the roots is -__________.

(viii) If α, β are the roots of $ax^2 + bx + c = 0$, then product of the roots is ________.

(ix) If α, β are the roots of $7x^2 - 5x + 3 = 0$, then sum of the roots is __________.

(x) If α, β are the roots of $5x^2 + 3x - 9 = 0$, then product of the roots is_________.

(xi) Far a quadratic equation $ax^2 + bx + c = 0$, $\frac{1}{\alpha\beta}$ is equal to _________.

(xii) Cube roots of unity are _________.

(xiii) Under usual notation sum of the cube roots of unity is __________.

(xiv) If 1, ω, ω^2 are the cube roots of unity, then ω^{-7} is equal to _________.

(xv) If α, β are the roots of the quadratic equation, then the quadratic equation is written as _________.

(xvi) If 2ω and $2\omega^2$ are the roots of an equation, the equation is _________.

Answers:

(i)	$b^2 - 4ac$	**(ii)**	equal	**(iii)**	real	**(iv)**	imaginary	**(v)**	rational
(vi)	irrational	**(vii)**	$-\frac{b}{a}$	**(viii)**	$\frac{c}{a}$	**(ix)**	$\frac{5}{7}$	**(x)**	$-\frac{9}{5}$
(xi)	$\frac{1}{\alpha\beta}$	**(xii)**	$1, \omega, \omega^2$	**(xiii)**	zero	**(xiv)**	ω^2	**(xv)**	$x^2 - (\alpha+\beta)x + \alpha\beta = 0$
(xvi)	$x^2 + 2x + 4 = 0$								

SUMMARY

✓**Discriminant** of the quadratic expression $ax^2 + bx + c$ is "$b^2 - 4ac$".

✓**The cube roots** of unity are $1, \frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

✓ **Complex cube roots** of unity are ω, ω^2.

✓ **Properties of cube roots of unity.**

(a) The product of three cube roots of unity is one. i.e., $(1)(\omega)(\omega^2)=\omega^3=1$.

(b) Each of the complex cube roots of unity is **reciprocal** of the other.

(c) Each of the complex cube roots of unity is the **square** of the other.

(d) The sum of all the cube roots of unity is zero, i.e., $1+\omega+\omega^2=0$.

✓ The **roots** of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are

$\alpha=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

✓ **The sum and the product** of the roots of $ax^2 + bx + c = 0$, $a \neq 0$ are

$\alpha+\beta=\dfrac{-b}{a}$ and $\alpha\beta=\dfrac{c}{a}$ respectively.

✓**Symmetric functions** of the roots of quadratic equation are those functions in which the roots involved are such that the values of the expressions remain unaltered, when roots are interchanged.

✓ **Formation** of a quadratic equation if its roots are given;

x^2 - (sum of the roots) x + product of the roots = 0

=> $x^2-(\alpha+\beta)x+\alpha\beta=0$

✓**Synthetic division** is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.

✓ A system of equations having a common solution is called a system of **simultaneous equations.**