Exercise 4.1

Resolve into partial fractions.

1) $\frac{7x-9}{(x+1)(x-3)}$

Solution:

$\frac{7x-9}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

Multiplying both sides by $(x+1)(x-3)$, we get

$7x-9 = A(x-3)+B(x+1)$ ………(1)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$7(-1)-9 = A(-1-3)+B(-1+1)$

$-16 = -4A$

$-4A = -16$

Dividing both sides by '-4', we get

$A = 4$

To find B, we put $x-3=0 \Rightarrow x=3$ in eq.(1), we get

$7(3)-9 = A(3-3)+B(3+1)$

$21-9 = A(0)+B(4)$

$12 = 4B$

$or\ 4B = 12$

Dividing both sides by '4', we get

$B = 3$

Thus required partial fraction are $\frac{4}{x+1}+\frac{3}{x-3}$

Hence, $\frac{7x-9}{(x+1)(x-3)} = \frac{4}{x+1}+\frac{3}{x-3}$

2) $\frac{x-11}{(x-4)(x+3)}$

Solution:

$\frac{x-11}{(x-4)(x+3)} = \frac{A}{x-4}+\frac{B}{x+3}$

Multiplying both sides by $(x-4)(x+3)$, we get

$x-11 = A(x+3)+B(x-4)$ ………(1)

To find A, we put $x-4=0 \Rightarrow x=4$ in eq.(1), we get

$4-11 = A(4+3)+B(4-4)$

$-7 = A(7)+B(0)$

$-7 = 7A$

$or\ 7A = -7$

Dividing both sides by '7', we get

$A = -1$

To find B, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$-3-11 = A(-3+3)+B(-3-4)$

$-14 = A(0)+B(-7)$

$-14 = -7B$

$or\ -7B = -14$

Dividing both sides by '-7', we get

$B = 2$

Thus required partial fraction are $\frac{-1}{x-4}+\frac{2}{x+3}$

Hence, $\frac{x-11}{(x-4)(x+3)} = -\frac{1}{x-4}+\frac{2}{x+3}$

3) $\frac{3x-1}{x^2-1}$

Solution:

$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x-1)(x+1)}$

$Let\ \frac{3x-1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1}$

Multiplying both sides by $(x-1)(x+1)$, we get

$3x-1 = A(x+1)+B(x-1)$ ………(1)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$3(1)-1 = A(1+1)+B(1-1)$

$3-1 = A(2)+B(0)$

$2 = 2A$

$or\ 2A = 2$

Dividing both sides by '2', we get

$A = 1$

To find B, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$3(-1)-1 = A(-1+1)+B(-1-1)$

$-3-1 = A(0)+B(-2)$

$-4 = -2B$

$or\ -2B = -4$

Dividing both sides by '-2', we get

$B = 2$

Thus required partial fraction are $\frac{1}{x-1}+\frac{2}{x+1}$

Hence, $\frac{3x-1}{x^2-1} = \frac{1}{x-1}+\frac{2}{x+1}$

4) $\frac{x-5}{x^2+2x-3}$

Solution:

$\frac{x-5}{x^2+2x-3} = \frac{x-5}{x^2+3x-x-3}$

$= \frac{x-5}{x(x+3)-1(x+3)} = \frac{x-5}{(x-1)(x+3)}$

$Let\ \frac{x-5}{(x-1)(x+3)} = \frac{A}{x-1}+\frac{B}{x+3}$

Multiplying both sides by $(x-1)(x+3)$, we get

$x-5 = A(x+3)+B(x-1)$ ………(1)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$1-5 = A(1+3)+B(1-1)$

$-4 = A(1+3)+B(0)$

$-4 = 4A$

$or\ 4A = -4$

Dividing both sides by '4', we get

$A = -1$

To find B, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$-3-5 = A(-3+3)+B(-3-1)$

$-8 = A(0)+B(-4)$

$-8 = -4B$

$or\ -4B = -8$

Dividing both sides by '-4', we get

$B = 2$

Thus required partial fraction are $\frac{-1}{x-1}+\frac{2}{x+3}$

Hence, $\frac{x-5}{x^2+2x-3} = -\frac{1}{x-1}+\frac{2}{x+3}$

5) $\frac{3x+3}{(x-1)(x+2)}$

Solution:

$Let\ \frac{3x+3}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2}$

Multiplying both sides by $(x-1)(x+2)$, we get

$3x+3 = A(x+2)+B(x-1)$ ………(1)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$3(1)+3 = A(1+2)+B(1-1)$

$3+3 = A(3)+B(0)$

$6 = 3A$

$or\ 3A = 6$

Dividing both sides by '3', we get

$A = 2$

To find B, we put $x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$3(-2)+3 = A(-2+2)+B(-2-1)$

$-6+3 = A(0)+B(-3)$

$-3 = -3B$

$or\ -3B = -3$

Dividing both sides by '-3', we get

$B = 1$

Thus required partial fraction are $\frac{2}{x-1}+\frac{1}{x+2}$

Hence, $\frac{3x+3}{(x-1)(x+2)} = \frac{2}{x-1}+\frac{1}{x+2}$

6) $\frac{7x-25}{(x-4)(x-3)}$

Solution:

$Let\ \frac{7x-25}{(x-4)(x-3)} = \frac{A}{x-4}+\frac{B}{x-3}$

Multiplying both sides by $(x-4)(x-3)$, we get

$7x-25 = A(x-3)+B(x-4)$ ………(1)

To find A, we put $x-4=0 \Rightarrow x=4$ in eq.(1), we get

$7(4)-25 = A(4-3)+B(4-4)$

$28-25 = A(1)+B(0)$

$3 = A$

$or\ A = 3$

To find B, we put $x-3=0 \Rightarrow x=3$ in eq.(1), we get

$7(3)-25 = A(3-3)+B(3-4)$

$21-25 = A(0)+B(-1)$

$-4 = -B$

$or\ -B = -4$

Dividing both sides by '-1', we get

$B = 4$

Thus required partial fraction are $\frac{3}{x-4}+\frac{4}{x-3}$

Hence, $\frac{7x-25}{(x-4)(x-3)} = \frac{3}{x-4}+\frac{4}{x-3}$

7) $\frac{x^2+2x+1}{(x-2)(x+3)}$

Solution:

$\frac{x^2+2x+1}{(x-2)(x+3)} = \frac{x^2+2x+1}{x^2+3x-2x-6}$

$= \frac{x^2+2x+1}{x^2+x-6}$

This is an improper fraction so by long division, we have

$$x^2+x-6 \overline{)\ x^2+2x+1\ }\ \text{quotient } 1$$

$$\underline{\pm x^2 \pm x \mp 6}$$

$$x+7$$

$\frac{x^2+2x+1}{(x-2)(x+3)} = 1+\frac{x+7}{x^2+x-6}$

$= 1+\frac{x+7}{(x-2)(x+3)}$

$Let\ \frac{x+7}{(x-2)(x+3)} = \frac{A}{x-2}+\frac{B}{x+3}$

Multiplying both sides by $(x-2)(x+3)$, we get

$x+7 = A(x+3)+B(x-2)$ ………(1)

To find A, we put $x-2=0 \Rightarrow x=2$ in eq.(1), we get

$2+7 = A(2+3)+B(2-2)$

$9 = A(5)+B(0)$

$9 = 5A$

$or\ 5A = 9$

Dividing both sides by '5', we get

$A = \frac{9}{5}$

To find B, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$-3+7 = A(-3+3)+B(-3-2)$

$4 = A(0)+B(-5)$

$4 = -5B$

$or\ -5B = 4$

Dividing both sides by '−5', we get

$B = -\frac{4}{5}$

Thus required partial fraction are $\frac{9/5}{x-2}+\frac{-4/5}{x+3}$

Hence, $\frac{x^2+2x+1}{(x-2)(x+3)} = 1+\frac{9}{5(x-2)}-\frac{4}{5(x+3)}$

8) $\frac{6x^3+5x^2-7}{3x^2-2x-1}$

Solution:

This an improper fraction so by long division, we have

$$3x^2-2x-1 \overline{)\ 6x^3+5x^2-7\ }\ \text{quotient } 2x+3$$

$$\underline{\pm 6x^3 \mp 4x^2 \mp 2x}$$

$$9x^2+2x-7$$

$$\underline{\pm 9x^2 \mp 6x \mp 3}$$

$$8x-4$$

$\frac{6x^3+5x^2-7}{3x^2-2x-1} = 2x+3+\frac{8x-4}{3x^2-2x-1}$

$= 2x+3+\frac{8x-4}{3x^2-3x+x-1}$

$= 2x+3+\frac{8x-4}{3x(x-1)+1(x-1)}$

$= 2x+3+\frac{8x-4}{(3x+1)(x-1)}$

$Let\ \frac{8x-4}{(3x+1)(x-1)} = \frac{A}{3x+1}+\frac{B}{x-1}$

Multiplying both sides by $(3x+1)(x-1)$, we get

$8x-4 = A(x-1)+B(3x+1)$ ………(1)

To find A, we put $3x+1=0 \Rightarrow 3x=-1 \Rightarrow x=-\frac{1}{3}$ in eq.(1), we get

$8\left(-\frac{1}{3}\right)-4 = A\left(-\frac{1}{3}-1\right)+B\left[3\left(-\frac{1}{3}\right)+1\right]$

$-\frac{8}{3}-4 = A\left(-\frac{4}{3}\right)+B(0)$

$-\frac{20}{3} = -\frac{4}{3}A$

$or -\frac{4}{3}A = -\frac{20}{3}$

$\frac{4}{3}A = \frac{20}{3}$

$A = \frac{20}{3}\times\frac{3}{4}$

$A = 5$

To find B, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$8(1)-4 = A(1-1)+B[3(1)+1]$

$8-4 = A(0)+B(4)$

$4 = 4B$

$or\ 4B = 4$

Dividing both sides by '4', we get

$B = 1$

Thus required partial fraction are $\frac{5}{3x+1}+\frac{1}{x-1}$

Hence, $\frac{6x^3+5x^2-7}{3x^2-2x-1} = 2x+3+\frac{5}{3x+1}+\frac{1}{x-1}$

Resolution of a fraction when D(x) consists of repeated linear factors:

Rule II:

If a linear factor (ax + b) occurs n times as a factor of D(x), then there are n partial fractions of the form.

$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+......+\frac{A_n}{(ax+b)^n}$ where $A_1, A_2.......A_n$ are constants and $n \geq 2$ is a positive integer.

$$\frac{N(x)}{D(x)}=\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+......+\frac{A_n}{(ax+b)^n}$$

Exercise 4.2

Resolve into partial fraction.

(1) $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

Let $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ ....(1)

$x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

$x^2-3x+1=Ax^2-3Ax+2A+Bx-2B+Cx^2-2Cx+C$

$x^2-3x+1=Ax^2+Cx^2-3Ax+Bx-2Cx+2A-2B+C$ ....(2)

To find C, we put $x-2=0 \Rightarrow x=2$ in eq.(1), we get

$(2)^2-3(2)+1=A(2-1)(2-2)+B(2-2)+C(2-1)^2$

$4-6+1=A(1)(0)+B(0)+C(1)^2$

$5-6=A(0)+B(0)+C$

$-1=C$

$or\ C=-1$

To find B, we put $(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ in eq.(1), we get

$(1)^2-3(1)+1=A(1-1)(1-2)+B(1-2)+C(1-1)^2$

$1-3+1=A(0)(-1)+B(-1)+C(0)$

$2-3=A(0)+B(-1)+C(0)$

$-1=-B$

$\Rightarrow B=1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=1$

$A+(-1)=1$

$A=1+1$

$A=2$

Thus required partial fractions are $\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{-1}{x-2}$

Hence, $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{2}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{x-2}$

(2) $\frac{x^2+7x+11}{(x+2)^2(x+3)}$

Solution:

Let $\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x+3}$

Multiplying both sides by $(x+2)^2(x+3)$, we get

$x^2+7x+11=A(x+2)(x+3)+B(x+3)+C(x+2)^2$ ....(1)

$x^2+7x+11=A(x^2+5x+6)+B(x+3)+C(x^2+4x+4)$

$x^2+7x+11=Ax^2+5Ax+6A+Bx+3B+Cx^2+4Cx+4C$

$x^2+7x+11=Ax^2+Cx^2+5Ax+Bx+4Cx+6A+3B+4C$ ....(2)

To find C, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$(-3)^2+7(-3)+11=A(-3+2)(-3+3)+B(-3+3)+C(-3+2)^2$

$9-21+11=A(-1)(0)+B(0)+C(-1)^2$

$20-21=A(0)+B(0)+C(1)$

$-1=C$

$or\ C=-1$

To find B, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$(-2)^2+7(-2)+11=A(-2+2)(-2+3)+B(-2+3)+C(-2+2)^2$

$4-14+11=A(0)(1)+B(1)+C(0)^2$

$15-14=A(0)+B(1)+C(0)$

$1=B$

$\Rightarrow B=1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=1$

$A+(-1)=1$

$A=1+1$

$A=2$

Thus required partial fractions are $\frac{2}{x+2}+\frac{1}{(x+2)^2}+\frac{-1}{x+3}$

Hence, $\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{2}{x+2}+\frac{1}{(x+2)^2}-\frac{1}{x+3}$

(3) $\frac{9}{(x-1)(x+2)^2}$

Solution:

Let $\frac{9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

Multiplying both sides by $(x-1)(x+2)^2$, we get

$9=A(x+2)^2+B(x-1)(x+2)+C(x-1)$ ....(1)

$9=A(x^2+4x+4)+B(x^2+x-2)+C(x-1)$

$9=Ax^2+4Ax+4A+Bx^2+Bx-2B+Cx-C$

$9=Ax^2+Bx^2+4Ax+Bx+Cx+4A-2B-C$ ....(2)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$9=A(1+2)^2+B(1-1)(1+2)+C(1-1)$

$9=A(3)^2+B(0)(3)+C(0)$

$9=A(9)+B(0)+C(0)$

$9=9A$

$or\ 9A=9$

Dividinge the both sides by '9', we get

$A=1$

To find C, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$9=A(-2+2)^2+B(-2-1)(-2+2)+C(-2-1)$

$9=A(0)^2+B(-3)(0)+C(-3)$

$9=A(0)+B(0)+C(-3)$

$9=-3C$

$or\ -3C=9$

Dividing both sides by '-3', we get

$C=-3$

To find B, equating coefficient of x^2 on both sides of (2), we get

$A+B=0$

$1+B=0$

$B=-1$

Thus required partial fractions are $\frac{1}{x-1}+\frac{-1}{x+2}+\frac{-3}{(x+2)^2}$

Hence, $\frac{9}{(x-1)(x+2)^2}=\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$

(4) $\frac{x^4+1}{x^2(x-1)}$

Solution:

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & x+1 \\ x^3-x^2 & x^4+1 \\ & \underline{\pm x^4 \mp x^3} \\ & x^3+1 \\ & \underline{\pm x^3 \mp x^2} \\ & x^2+1 \end{array}$$

$\frac{x^4+1}{x^2(x-1)}=x+1+\frac{x^2+1}{x^2(x-1)}$

Let $\frac{x^2+1}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$

Multiplying both sides by $x^2(x-1)$, we get

$x^2+1=A(x)(x+1)+B(x-1)+Cx^2$ ....(1)

$x^2+1=Ax^2-Ax+Bx-B+Cx^2$

$x^2+1=Ax^2+Cx^2-Ax+Bx-B$ ....(2)

To find C, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$(1)^2+1=A(1)(1-1)+B(1-1)+C(1)^2$

$1+1=A(1)(0)+B(0)+C(1)$

$2=A(0)+B(0)+C(1)$

$2=C$

$or\ C=2$

To find B, we put $x^2=0 \Rightarrow x=0$ in eq.(1), we get

$(0)^2+1=A(0)(0-1)+B(0-1)+C(0)^2$

$1=A(0)(-1)+B(-1)+C(0)$

$1=-B$

$or\ B=-1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=1$

$A+2=1$

$A=1-2$

$A=-1$

Thus required partial fractions are $\frac{-1}{x}+\frac{-1}{x^2}+\frac{2}{x-1}$

Hence, $\frac{x^4+1}{x^2(x-1)}=x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x-1}$

(5) $\frac{7x+4}{(3x+2)(x+1)^2}$

Solution:

Let $\frac{7x+4}{(3x+2)(x+1)^2}=\frac{A}{(3x+2)}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

Multiplying both sides by $(3x+2)(x+1)^2$, we get

$7x+4=A(x+1)^2+B(3x+2)(x+1)+C(3x+2)$ ....(1)

$7x+4=A(x^2+2x+1)+B(3x^2+5x+2)+C(3x+2)$

$7x+4=Ax^2+2Ax+A+3Bx^2+5Bx+2B+3Cx+2C$

$7x+4=Ax^2+3Bx^2+2Ax+5Bx+3Cx+A+2B+2C$ ....(2)\

To find A, we put $3x+2=0 \Rightarrow 3x=-2 \Rightarrow x=\frac{-2}{3}$ in eq.(1), we get

$7\left(\frac{-2}{3}\right)+4=A\left(\frac{-2}{3}+1\right)^2+B\left(3\left(\frac{-2}{3}\right)+2\right)\left(\frac{-2}{3}+1\right)+C\left(3\left(\frac{-2}{3}\right)+2\right)$

$-\frac{14}{3}+4=A\left(\frac{1}{3}\right)^2+B(-2+2)\left(\frac{1}{3}\right)+C(-2+2)$

$-\frac{2}{3}=A\left(\frac{1}{9}\right)+B(0)+C(0)$

$-\frac{2}{3}=\frac{1}{9}A$

$or\ \frac{1}{9}A=-\frac{2}{3}$

$A=-\frac{2}{3}\times\frac{9}{1}$

$A=-6$

To find C, we put $(x+1)^2=0 \Rightarrow x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$7(-1)+4=A(-1+1)^2+B(3(-1)+2)(-1+1)+C(3(-1)+2)$

$-7+4=A(0)^2+B(-3+2)(0)+C(-3+2)$

$-3=A(0)+B(0)+C(-1)$

$-3=-C$

$or -C=-3$

$\Rightarrow C=3$

To find B, equating coefficient of x^2 on both sides of (2), we get

$A+3B=0$

$-6+3B=0$

$3B=6$

$\Rightarrow B=2$

Thus required partial fractions are $\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

Hence, $\frac{7x+4}{(3x+2)(x+1)^2}=\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

(6) $\frac{1}{(x-1)^2(x+1)}$

Solution:

Let $\frac{1}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

Multiplying both sides by $(x-1)^2(x+1)$, we get

$1=A(x-1)(x+1)+B(x+1)+C(x-1)^2$ ....(1)

$1=A(x^2-1)+B(x+1)+C(x^2-2x+1)$

$1=Ax^2-A+Bx+B+Cx^2-2Cx+C$

$1=Ax^2+Cx^2+Bx-2Cx-A+B+C$ ....(2)

To find C, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$1=A(-1-1)(-1+1)+B(-1+1)+C(-1-1)^2$

$1=A(-2)(0)+B(0)+C(-2)^2$

$1=A(0)+B(0)+C(4)$

$1=4C$

$or\ 4C=1$

$\Rightarrow C=\frac{1}{4}$

To find B, we put $(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ in eq.(1), we get

$1=A(1-1)(1+1)+B(1+1)+C(1-1)^2$

$1=A(0)(2)+B(2)+C(0)^2$

$1=A(0)+B(2)+C(0)$

$1=2B$

$or\ 2B=1$

$\Rightarrow B=\frac{1}{2}$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=0$

$A+\frac{1}{4}=0$

$\Rightarrow A=-\frac{1}{4}$

Thus required partial fractions are $\frac{-1/4}{x-1}+\frac{1/2}{(x-1)^2}+\frac{1/4}{x+1}$

Hence, $\frac{1}{(x-1)^2(x+1)}=\frac{-1}{4(x-1)}+\frac{1}{2(x-1)^2}+\frac{1}{4(x+1)}$

(7) $\frac{3x^2+15x+16}{(x+2)^2}$

Solution:

$=\frac{3x^2+15x+16}{x^2+4x+4}$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & 3 \\ x^2+4x+4 & 3x^2+15x+16 \\ & \underline{\pm 3x^2 \pm 12x \pm 12} \\ & 3x+4 \end{array}$$

$\frac{3x^2+15x+16}{x^2+4x+4}=3+\frac{3x+4}{(x+2)^2}$

Let $\frac{3x+4}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$

Multiplying both sides by $(x+2)^2$, we get

$3x+4=A(x+2)+B$ ....(1)

$3x+4=Ax+2A+B$ ....(2)

To find B, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$3(-2)+4=A(-2+2)+B$

$-6+4=A(0)+B$

$-2=B$

$or\ B=-2$

To find A, equating coefficient of x on both sides of (2), we get

$A=3$

Thus required partial fractions are $\frac{3}{x+2}+\frac{-2}{(x+2)^2}$

Hence, $\frac{3x^2+15x+16}{(x+2)^2}=3+\frac{3}{x+2}-\frac{2}{(x+2)^2}$

(8) $\frac{1}{(x^2-1)(x+1)}$

Solution:

$=\frac{1}{(x-1)(x+1)(x+1)}$

$=\frac{1}{(x-1)(x+1)^2}$

Let $\frac{1}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

Multiplying both sides by $(x-1)(x+1)^2$, we get

$1=A(x+1)^2+B(x-1)(x+1)+C(x-1)$ ....(1)

$1=A(x^2+2x+1)+B(x^2-1)+C(x-1)$

$1=Ax^2+2Ax+A+Bx^2-B+Cx-C$

$1=Ax^2+Bx^2+2Ax+Cx+A-B-C$ ....(2)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$1=A(1+1)^2+B(1-1)(1+1)+C(1-1)$

$1=A(2)^2+B(0)(2)+C(0)$

$1=A(4)+B(0)+C(0)$

$1=4A$

$or\ 4A=1$

$\Rightarrow A=\frac{1}{4}$

To find C, we put $(x+1)^2=0 \Rightarrow x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$1=A(-1+1)^2+B(-1-1)(-1+1)+C(-1-1)$

$1=A(0)^2+B(-2)(0)+C(-2)$

$1=A(0)+B(0)+C(-2)$

$1=-2C$

$or\ -2C=1$

$\Rightarrow C=-\frac{1}{2}$

To find B, equating coefficient of x^2 on both sides of (2), we get

$A+B=0$

$\frac{1}{4}+B=0$

$\Rightarrow B=-\frac{1}{4}$

Thus required partial fractions are $\frac{1/4}{x-1}+\frac{-1/4}{x+1}+\frac{-1/2}{(x+1)^2}$

Hence, $\frac{1}{(x^2-1)(x+1)}=\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x+1)^2}$

Resolution of fraction when D (x) consists of non-repeated irreducible quadratic factors.

Rule III:

If a quadratic factor (ax² + bx + c) with a ≠ 0 occur once as a factor of D(x), the partial fraction is of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be found.

Exercise 4.3

Resolve into partial fractions.

(1) $\frac{3x-11}{(x+3)(x^2+1)}$

Solution:

$$\frac{3x-11}{(x+3)(x^2+1)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$3x-11=A(x^2+1)+(Bx+C)(x+3)$ ....(1)

$3x-11=Ax^2+A+Bx^2+3Bx+Cx+3C$

$3x-11=Ax^2+Bx^2+3Bx+Cx+A+3C$ ....(2)

To find A, we put $x+3=0\Rightarrow x=-3$ in eq.(1), we get

$3(-3)-11=A((-3)^2+1)+(B(-3)+C)(-3+3)$

$-9-11=A(9+1)+(-3B+C)(0)$

$-20=10A$

$or\ 10A=-20$

Dividing both sides by '10', we get

$\Rightarrow A=-2$

To find B and C, equating coefficient of x^2 and constant on both sides of eq.(2), we get

$A+B=0$

$-2+B=0$

$\Rightarrow B=2$

And $A+3C=-11$

$-2+3C=-11$

$3C=-11+2$

$3C=-9$

Dividing both sides by '3', we get

$C=-3$

Thus required partial fractions are $\frac{-2}{x+3}+\frac{2x+(-3)}{x^2+1}$

Hence, $\frac{3x-11}{(x+3)(x^2+1)}=\frac{-2}{x+3}+\frac{2x-3}{x^2+1}$

(2) $\frac{3x+7}{(x^2+1)(x+3)}$

Solution:

$$\frac{3x+7}{(x^2+1)(x+3)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+3}$$

Multiplying both sides by $(x^2+1)(x+3)$, we get

$3x+7=(Ax+B)(x+3)+C(x^2+1)$ ....(1)

$3x+7=Ax^2+3Ax+Bx+3B+Cx^2+C$

$3x+7=Ax^2+Cx^2+3Ax+Bx+3B+C$ ....(2)

To find C, we put $x+3=0\Rightarrow x=-3$ in eq.(1), we get

$3(-3)+7=(A(-3)+B)(-3+3)+C((-3)^2+1)$

$-9+7=(-3A+B)(0)+C(9+1)$

$-2=10C$

$or\ 10C=-2$

Dividing both sides by '10', we get

$\Rightarrow C=-\frac{1}{5}$

To find A and B, equating coefficient of x^2 and constant on both sides of (2), we get

$A+C=0$

$A+\left(-\frac{1}{5}\right)=0$

$\Rightarrow A=\frac{1}{5}$

And $3B+C=7$

$3B+\left(-\frac{1}{5}\right)=7$

$3B=7+\frac{1}{5}$

$3B=\frac{36}{5}$

$B=\frac{36}{5}\times\frac{1}{3}$

$B=\frac{12}{5}$

Thus required partial fractions are $\frac{\frac{1}{5}x+\frac{12}{5}}{x^2+1}+\frac{\frac{-1}{5}}{x+3}$

Hence, $\frac{3x+7}{(x^2+1)(x+3)}=\frac{x+12}{5(x^2+1)}-\frac{1}{5(x+3)}$

(3) $\frac{1}{(x+1)(x^2+1)}$

Solution:

$$\frac{1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+1)(x^2+1)$, we get

$1=A(x^2+1)+(Bx+C)(x+1)$ ....(1)

$1=Ax^2+A+Bx^2+Bx+Cx+C$

$1=Ax^2+Bx^2+Bx+Cx+A+C$ ....(2)

To find A, we put $x+1=0\Rightarrow x=-1$ in eq.(1), we get

$1=A((-1)^2+1)+(B(-1)+C)(-1+1)$

$1=A(1+1)+(-B+C)(0)$

$1=A(2)$

$or\ 2A=1$

Dividing both sides by '2', we get

$\Rightarrow A=\frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=0$

$\frac{1}{2}+B=0$ $\because A=\frac{1}{2}$

$\Rightarrow B=-\frac{1}{2}$

And $A+C=1$

$\frac{1}{2}+C=1$ $\because A=\frac{1}{2}$

$C=1-\frac{1}{2}$

$C=\frac{1}{2}$

Thus required partial fractions are $\frac{\frac{1}{2}}{x+1}+\frac{\frac{-1}{2}x+\frac{1}{2}}{x^2+1}$

Hence, $\frac{1}{(x+1)(x^2+1)}=\frac{1}{2(x+1)}-\frac{x-1}{2(x^2+1)}$

(4) $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$9x-7=A(x^2+1)+(Bx+C)(x+3)$ ....(1)

$9x-7=Ax^2+A+Bx^2+3Bx+Cx+3C$

$9x-7=Ax^2+Bx^2+3Bx+Cx+A+3C$ ....(2)

To find A, we put $x+3=0\Rightarrow x=-3$ in eq.(1), we get

$9(-3)-7=A((-3)^2+1)+(B(-3)+C)(-3+3)$

$-27-7=A(9+1)+(-3B+C)(0)$

$-34=10A$

$or\ 10A=-34$

Dividing both sides by '10', we get

$\Rightarrow A=\frac{-34}{10}=-\frac{17}{5}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=0$

$-\frac{17}{5}+B=0$ $\because A=-\frac{17}{5}$

$\Rightarrow B=\frac{17}{5}$

And $A+3C=-7$

$-\frac{17}{5}+3C=-7$ $\because A=-\frac{17}{5}$

$3C=-7+\frac{17}{5}$

$3C=-\frac{18}{5}$

$C=-\frac{18}{5}\times\frac{1}{3}$

$C=-\frac{6}{5}$

Thus required partial fractions are $\frac{\frac{-17}{5}}{x+1}+\frac{\frac{17}{5}x-\frac{6}{5}}{x^2+1}$

Hence, $\frac{9x-7}{(x+3)(x^2+1)}=\frac{-17}{5(x+1)}+\frac{17x-6}{5(x^2+1)}$

(5) $\frac{3x+7}{(x+3)(x^2+4)}$

Solution:

Let $\frac{3x+7}{(x+3)(x^2+4)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+3)(x^2+4)$, we get

$3x+7=A(x^2+4)+(Bx+C)(x+3)$ ....(1)

$3x+7=Ax^2+4A+Bx^2+3Bx+Cx+3C$

$3x+7=Ax^2+Bx^2+3Bx+Cx+4A+3C$ ....(2)

To find A, we put $x+3=0\Rightarrow x=-3$ in eq.(1), we get

$3(-3)+7=A((-3)^2+4)+(B(-3)+C)(-3+3)$

$-9+7=A(9+4)+(-3B+C)(0)$

$-2=13A$

$or\ 13A=-2$

Dividing both sides by '13', we get

$\Rightarrow A=-\frac{2}{13}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=0$

$-\frac{2}{13}+B=0$ $\because A=-\frac{2}{13}$

$\Rightarrow B=\frac{2}{13}$

And $4A+3C=7$

$4\left(-\frac{2}{13}\right)+3C=7$

$3C=7+\frac{8}{13}$

$3C=\frac{99}{13}$

$C=\frac{33}{13}$

Thus required partial fractions are $\frac{\frac{-2}{13}}{x+3}+\frac{\frac{2}{13}x+\frac{33}{13}}{x^2+4}$

Hence, $\frac{3x+7}{(x+3)(x^2+4)}=\frac{-2}{13(x+3)}+\frac{2x+33}{13(x^2+4)}$

(6) $\frac{x^2}{(x+2)(x^2+4)}$

Solution:

Let $\frac{x^2}{(x+2)(x^2+4)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+2)(x^2+4)$, we get

$x^2=A(x^2+4)+(Bx+C)(x+2)$ ....(1)

$x^2=Ax^2+4A+Bx^2+2Bx+Cx+2C$

$x^2=Ax^2+Bx^2+2Bx+Cx+4A+2C$ ....(2)

To find A, we put $x+2=0\Rightarrow x=-2$ in eq.(1), we get

$(-2)^2=A((-2)^2+4)+(B(-2)+C)(-2+2)$

$4=A(4+4)+(-2B+C)(0)$

$4=8A$

Dividing both sides by '8', we get

$\Rightarrow A=\frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=1$

$\frac{1}{2}+B=1$ $\because A=\frac{1}{2}$

$B=1-\frac{1}{2}$

$\Rightarrow B=\frac{1}{2}$

And $4A+2C=0$

$4\left(\frac{1}{2}\right)+2C=0$ $\because A=\frac{1}{2}$

$2+2C=0$

$2C=-2$

$\Rightarrow C=-1$

Thus required partial fractions are $\frac{\frac{1}{2}}{x+2}+\frac{\frac{1}{2}x-1}{x^2+4}$

Hence, $\frac{x^2}{(x+2)(x^2+4)}=\frac{1}{2(x+2)}+\frac{x-2}{2(x^2+4)}$

(7) $\frac{1}{x^3+1}$

Solution:

$$\frac{1}{x^3+1}=\frac{1}{(x)^3+(1)^3}=\frac{1}{(x+1)(x^2-x+1)}$$

Let $\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$1=A(x^2-x+1)+(Bx+C)(x+1)$ ....(1)

$1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$

$1=Ax^2+Bx^2-Ax+Bx+Cx+A+C$ ....(2)

To find A, we put $x+1=0\Rightarrow x=-1$ in eq.(1), we get

$1=A((-1)^2-(-1)+1)+(B(-1)+C)(-1+1)$

$1=A(1+1+1)+(-B+C)(0)$

$1=A(3)+(-B+C)(0)$

$or\ 3A=1$

Dividing both sides by '3', we get

$\Rightarrow A=\frac{1}{3}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=0$

$\frac{1}{3}+B=0$ $\because A=\frac{1}{3}$

$B=-\frac{1}{3}$

And $A+C=1$

$\frac{1}{3}+C=1$ $\because A=\frac{1}{2}$

$C=1-\frac{1}{3}$

$C=\frac{2}{3}$

Thus required partial fractions are $\frac{\frac{1}{3}}{x+1}+\frac{\frac{-1}{3}x+\frac{2}{3}}{x^2-x+1}$

Hence, $\frac{1}{x^3+1}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$

(8) $\frac{x^2+1}{x^3+1}$

Solution:

$$\frac{x^2+1}{x^3+1}=\frac{x^2+1}{(x)^3+(1)^3}=\frac{x^2+1}{(x+1)(x^2-x+1)}$$

Let $\frac{x^2+1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$x^2+1=A(x^2-x+1)+(Bx+C)(x+1)$ ....(1)

$x^2+1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$

$x^2+1=Ax^2+Bx^2-Ax+Bx+Cx+A+C$ ....(2)

To find A, we put $x+1=0\Rightarrow x=-1$ in eq.(1), we get

$(-1)^2+1=A((-1)^2-(-1)+1)+(B(-1)+C)(-1+1)$

$1+1=A(1+1+1)+(-B+C)(0)$

$2=A(3)+(-B+C)(0)$

$2=A(3)$

$or\ 3A=2$

$\Rightarrow A=\frac{2}{3}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=1$

$\frac{2}{3}+B=1$ $\because A=\frac{2}{3}$

$B=1-\frac{2}{3}$

$B=\frac{1}{3}$

And $A+C=1$

$\frac{2}{3}+C=1$ $\because A=\frac{2}{3}$

$C=1-\frac{2}{3}$

$C=\frac{1}{3}$

Thus required partial fractions are $\frac{\frac{2}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}$

Hence, $\frac{x^2+1}{x^3+1}=\frac{2}{3(x+1)}+\frac{x+1}{3(x^2-x+1)}$

Resolution of a fraction when D(x) has repeated irreducible quadratic factors.

Rule IV

If a quadratic factor (ax^2+bx+c) with $a \neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}$$

The constants A, B, C and D are found in the usual way.

Exercise 4.4

Resolve into partial fractions.

(1) $\dfrac{x^3}{(x^2+4)^2}$

Solution:

Let $\dfrac{x^3}{(x^2+4)^2}=\dfrac{Ax+B}{x^2+4}+\dfrac{Cx+D}{(x^2+4)^2}$

Multiplying both sides by $(x^2+4)^2$, we get

$x^3=(Ax+B)(x^2+4)+(Cx+D)$ ....(1)

$x^3=Ax^3+4Ax+Bx^2+4B+Cx+D$ ....(2)

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(2)

We get

Coefficient of x^3: $A=1$

Coefficient of x^2: $B=0$

Coefficient of x: $4A+C=0$ ....(3)

Constant: $4B+D=0$ ....(4)

Put $A=1$ in eq.(3), we get

$4(1)+C=0$

$C=-4$

Put $B=0$ in eq.(4), we get

$4(0)+D=0$

$D=0$

Thus required partial fractions are $\dfrac{(1)x+0}{x^2+4}+\dfrac{(-4)x+0}{(x^2+4)^2}$

Hence, $\dfrac{x^3}{(x^2+4)^2}=\dfrac{x}{x^2+4}-\dfrac{4x}{(x^2+4)^2}$

(2) $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

Let $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

Multiplying both sides by $(x+1)(x^2+1)^2$, we get

$x^4+3x^2+x+1=A(x^2+1)^2+(Bx+C)(x^2+1)(x+1)+(Dx+E)(x+1)$ ....(1)

$x^4+3x^2+x+1=A(x^4+2x^2+1)+(Bx+C)(x^3+x^2+x+1)+Dx^2+Dx+Ex+E$

$x^4+3x^2+x+1=Ax^4+2Ax^2+A+Bx^4+Bx^3+Bx^2+Bx+Cx^3+Cx^2+Cx+C+Dx^2+Dx+Ex+E$

$x^4+3x^2+x+1=Ax^4+Bx^4+Bx^3+Cx^3+2Ax^2+Bx^2+Cx^2+Dx^2+Bx+Cx+Dx+Ex+A+C+E$ ....(2)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$(-1)^4+3(-1)^2+(-1)+1=A((-1)^2+1)^2$

$1+3-1+1=A(2)^2$

$4=4A$

$A=1$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4: $A+B=1$ ....(3)

Coefficient of x^3: $B+C=0$ ....(4)

Coefficient of x^2: $2A+B+C+D=3$ ....(5)

Coefficient of x: $B+C+D+E=1$ ....(6)

Put $A=1$ in eq.(3), we get

$1+B=1$

$B=0$

Put $B=0$ in eq.(4), we get

$0+C=0$

$C=0$

Put $A=1, B=0, C=0$ in eq.(5), we get

$2(1)+0+0+D=3$

$D=1$

Put $B=0, C=0, D=1$ in eq.(6), we get

$0+0+1+E=1$

$E=0$

Thus required partial fractions are $\dfrac{1}{x+1}+\dfrac{(0)x+0}{x^2+1}+\dfrac{(1)x+0}{(x^2+1)^2}$

Hence, $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}=\dfrac{1}{x+1}+\dfrac{x}{(x^2+1)^2}$

(3) $\dfrac{x^2}{(x+1)(x^2+1)^2}$

Solution:

Let $\dfrac{x^2}{(x+1)(x^2+1)^2}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

Multiplying both sides by $(x+1)(x^2+1)^2$, we get

$x^2=A(x^2+1)^2+(Bx+C)(x^2+1)(x+1)+(Dx+E)(x+1)$ ....(1)

$x^2=A(x^4+2x^2+1)+(Bx+C)(x^3+x^2+x+1)$
$\quad +Dx^2+Dx+Ex+E$

$x^2=Ax^4+2Ax^2+A+Bx^4+Bx^3+Bx^2+Bx+Cx^3$
$\quad +Cx^2+Cx+C+Dx^2+Dx+Ex+E$

$x^2=Ax^4+Bx^4+Bx^3+Cx^3+2Ax^2+Bx^2+Cx^2+Dx^2$
$\quad +Bx+Cx+Dx+Ex+A+C+E$ ....(2)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$(-1)^2=A((-1)^2+1)^2$

$1=A(1+1)^2$

$1=A(2)^2$

$1=4A$

or $4A=1$

$\Rightarrow A=\dfrac{1}{4}$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4: $A+B=0$ ....(3)

Coefficient of x^3: $B+C=0$ ....(4)

Coefficient of x^2: $2A+B+C+D=1$ ....(5)

Coefficient of x: $B+C+D+E=0$ ....(6)

Put $A=\dfrac{1}{4}$ in eq.(4), we get

$\dfrac{1}{4}+B=0$

$B=-\dfrac{1}{4}$

Put $B=-\dfrac{1}{4}$ in eq.(4), we get

$-\dfrac{1}{4}+C=0$

$C=\dfrac{1}{4}$

Put $A=\dfrac{1}{4}, B=-\dfrac{1}{4}, C=\dfrac{1}{4}$ in eq.(5), we get

$2\left(\dfrac{1}{4}\right)+\left(-\dfrac{1}{4}\right)+\left(\dfrac{1}{4}\right)+D=1$

$\dfrac{2}{4}-\dfrac{1}{4}+\dfrac{1}{4}+D=1$

$\dfrac{1}{2}+D=1$

$D=1-\dfrac{1}{2}$

$D=\dfrac{1}{2}$

Put $B=-\dfrac{1}{4}, C=\dfrac{1}{4}, D=\dfrac{1}{2}$ in eq.(6), we get

$\left(-\dfrac{1}{4}\right)+\left(\dfrac{1}{4}\right)+\left(\dfrac{1}{2}\right)+E=0$

$-\dfrac{1}{4}+\dfrac{1}{4}+\dfrac{1}{2}+E=0$

$E=-\dfrac{1}{2}$

Thus required partial fractions are $\dfrac{\frac{1}{4}}{x+1}+\dfrac{\left(\frac{-1}{4}\right)x+\left(\frac{1}{4}\right)}{x^2+1}+\dfrac{\left(\frac{1}{2}\right)x+\left(\frac{-1}{2}\right)}{(x^2+1)^2}$

Hence, $\dfrac{x^2}{(x+1)(x^2+1)^2}=\dfrac{1}{4(x+1)}-\dfrac{x-1}{4(x^2+1)}+\dfrac{x-1}{2(x^2+1)^2}$

(4) $\dfrac{x^2}{(x-1)(x^2+1)^2}$

Solution:

Let $\dfrac{x^2}{(x-1)(x^2+1)^2}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

Multiplying both sides by $(x-1)(x^2+1)^2$, we get

$x^2=A(x^2+1)^2+(Bx+C)(x^2+1)(x-1)+(Dx+E)(x-1)$ ....(1)

$x^2=A(x^4+2x^2+1)+(Bx+C)(x^3-x^2+x-1)$
$\quad +Dx^2-Dx+Ex-E$

$x^2=Ax^4+2Ax^2+A+Bx^4-Bx^3+Bx^2-Bx+Cx^3$
$\quad -Cx^2+Cx-C+Dx^2-Dx+Ex-E$

$x^2=Ax^4+Bx^4-Bx^3+Cx^3+2Ax^2+Bx^2-Cx^2+Dx^2$
$\quad -Bx+Cx-Dx+Ex+A-C-E$ ....(2)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$(1)^2=A((1)^2+1)^2$

$1=A(1+1)^2$

$1=A(2)^2$

$1=4A$

or $4A=1$

$\Rightarrow A=\dfrac{1}{4}$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4: $A+B=0$ ....(3)

Coefficient of x^3: $-B+C=0$ ....(4)

Coefficient of x^2: $2A+B-C+D=1$ ....(5)

Coefficient of x: $-B+C-D+E=0$ ....(6)

Put $A=\dfrac{1}{4}$ in eq.(4), we get

$\dfrac{1}{4}+B=0$

$B=-\dfrac{1}{4}$

Put $B=-\dfrac{1}{4}$ in eq.(4), we get

$-\left(-\dfrac{1}{4}\right)+C=0$

$\dfrac{1}{4}+C=0$

$C=-\dfrac{1}{4}$

Put $A=\dfrac{1}{4}, B=-\dfrac{1}{4}, C=-\dfrac{1}{4}$ in eq.(5), we get

$2\left(\dfrac{1}{4}\right)+\left(-\dfrac{1}{4}\right)-\left(-\dfrac{1}{4}\right)+D=1$

$\dfrac{2}{4}-\dfrac{1}{4}+\dfrac{1}{4}+D=1$

$\dfrac{1}{2}+D=1$

$D=1-\dfrac{1}{2}$

$D=\dfrac{1}{2}$

Put $B=-\dfrac{1}{4}, C=-\dfrac{1}{4}, D=\dfrac{1}{2}$ in eq.(6), we get

$-\left(-\dfrac{1}{4}\right)+\left(-\dfrac{1}{4}\right)-\dfrac{1}{2}+E=0$

$\dfrac{1}{4}-\dfrac{1}{4}-\dfrac{1}{2}+E=0$

$E=\dfrac{1}{2}$

Thus required partial fractions are $\dfrac{\frac{1}{4}}{x-1}+\dfrac{\left(\frac{-1}{4}\right)x+\left(\frac{-1}{4}\right)}{x^2+1}+\dfrac{\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)}{(x^2+1)^2}$

Hence, $\dfrac{x^2}{(x-1)(x^2+1)^2}=\dfrac{1}{4(x-1)}-\dfrac{x+1}{4(x^2+1)}+\dfrac{x+1}{2(x^2+1)^2}$

(5) $\dfrac{x^4}{(x^2+2)^2}$

Solution:

$\dfrac{x^4}{(x^2+2)^2}=\dfrac{x^4}{x^4+4x^2+4}$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & 1 \\ x^4+4x^2+4 & x^4 \\ & \underline{\pm x^4 \pm 4x^2 \pm 4} \\ & -4x^2-4 \\ & -(4x^2+4) \end{array}$$

$\dfrac{x^4}{(x^2+2)^2}=1-\dfrac{4x^2+4}{(x^2+2)^2}$

Let $\dfrac{4x^2+4}{(x^2+2)^2}=\dfrac{Ax+B}{x^2+2}+\dfrac{Cx+D}{(x^2+2)^2}$

Multiplying both sides by $(x^2+2)^2$, we get

$4x^2+4=(Ax+B)(x^2+2)+(Cx+D)$

$4x^2+4=Ax^3+2Ax+Bx^2+2B+Cx+D$

$4x^2+4=Ax^3+Bx^2+2Ax+Cx+2B+D$ ....(1)

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(1), We get

Coefficient of x^3: $A=0$

Coefficient of x^2: $B=4$

Coefficient of x: $2A+C=0$ ....(2)

Constant: $2B+D=4$ ....(3)

Put $A=0$ in eq.(2), we get

$2(0)+C=0$

$C=0$

Put $B=4$ in eq.(3), we get

$2(4)+D=4$

$8+D=4$

$D=4-8$

$D=-4$

Thus required partial fractions are $\dfrac{(0)x+(4)}{x^2+2}+\dfrac{(0)x+(-4)}{(x^2+2)^2}$

Hence, $\dfrac{x^4}{(x^2+2)^2}=1-\left[\dfrac{4}{x^2+2}-\dfrac{4}{(x^2+2)^2}\right]$

$\qquad =1-\dfrac{4}{x^2+2}+\dfrac{4}{(x^2+2)^2}$

(6) $\dfrac{x^5}{(x^2+1)^2}$

Solution:

$\dfrac{x^5}{(x^2+1)^2}=\dfrac{x^5}{x^4+2x^2+1}$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & x \\ x^4+2x^2+1 & x^5 \\ & \underline{\pm x^5 \pm 2x^3 \pm x} \\ & -2x^3-x \\ & -(2x^3+x) \end{array}$$

$\dfrac{x^5}{(x^2+1)^2}=x-\dfrac{2x^3+x}{(x^2+1)^2}$

Let $\dfrac{2x^3+x}{(x^2+1)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$

Multiplying both sides by $(x^2+1)^2$, we get

$2x^3+x=(Ax+B)(x^2+1)+(Cx+D)$

$2x^3+x=Ax^3+Ax+Bx^2+B+Cx+D$

$2x^3+x=Ax^3+Bx^2+Ax+Cx+B+D$ ....(1)

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(1), We get

Coefficient of x^3: $A=2$

Coefficient of x^2: $B=0$

Coefficient of x: $A+C=1$ ....(2)

Constant: $B+D=0$ ....(3)

Put $A=2$ in eq.(2), we get

$A+C=1$

$2+C=1$

$C=1-2$

$C=-1$

Put $B=0$ in eq.(3), we get

$0+D=0$

$D=0$

Thus required partial fractions are $\dfrac{2x+0}{x^2+1}+\dfrac{(-1)x+0}{(x^2+1)^2}$

Hence, $\dfrac{x^5}{(x^2+1)^2}=x-\left[\dfrac{2x}{x^2+1}+\dfrac{-x}{(x^2+1)^2}\right]$

Miscellaneous Exercise 4

Q1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The identity $(5x+4)^2 = 25x^2 + 40x + 16$ is true for

(a) one value of x (b) two values of x

(c) all values of x (d) none of these

(ii) A function of the form $f(x) = \frac{N(x)}{D(x)}$, with $D(x) \neq 0$, where N(x) and D(x) are polynomials in x is called

(a) an identity (b) an equation

(c) a fraction (d) none of these

(iii) A fraction in which the degree of the numerator is greater or equal the degree of denominator is called

(a) a proper fraction (b) an improper fraction

(c) an equation (d) algebraic relation

(iv) A fraction in which the degree of numerator is less than the degree of the denominator is called

(a) an equation (b) an improper fraction

(c) an identity (d) a proper fraction

(v) $\frac{2x+1}{(x+1)(x-1)}$ is

(a) an improper fraction (b) an equation

(c) a proper fraction (d) none of these

(vi) $1 - \frac{x^2}{x^2+1}$ is

(a) a linear equation (b) an equation

(c) an identity (d) none of these

(vii) $\frac{x^2+1}{(x+1)(x+2)}$ is

(a) a proper fraction (b) an improper fraction

(c) an identity (d) a constant term

(viii) Partial fractions of $\frac{x-2}{(x-1)(x+2)}$ are of the form

(a) $\frac{A}{x-1} + \frac{B}{x+2}$ (b) $\frac{Ax}{x-1} + \frac{B}{x+2}$

(c) $\frac{A}{x-1} + \frac{Bx+C}{x+2}$ (d) $\frac{Ax+B}{x-1} + \frac{C}{x+2}$

(ix) Partial fractions of $\frac{x+2}{(x+1)(x^2+2)}$ are of the form

(a) $\frac{A}{x+1} + \frac{B}{x^2+2}$ (b) $\frac{A}{x+1} + \frac{Bx+C}{x^2+2}$

(c) $\frac{Ax+B}{x+1} + \frac{C}{x^2+2}$ (d) $\frac{A}{x+1} + \frac{Bx}{x^2+2}$

(x) Partial fractions of $\frac{x^2+1}{(x+1)(x-1)}$ are of the form

(a) $\frac{A}{x+1} + \frac{B}{x-1}$ (b) $1 + \frac{A}{x+1} + \frac{B}{x-1}$

(c) $1 + \frac{Ax+B}{x+1} + \frac{C}{x-1}$ (d) $\frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

Answers:

(i)	[illegible]	(ii)	[illegible]	(iii)	[illegible]	(iv)	[illegible]	(v)	[illegible]
(vi)	[illegible]	(vii)	[illegible]	(viii)	[illegible]	(ix)	[illegible]	(x)	[illegible]

Q2. Write short answers of the following questions.

(i) Define a rational fraction.

Ans: Rational Fraction

An expression of the form $\frac{N(x)}{D(x)}$ with $D(x) \neq 0$ is called a rational fraction.

(ii) What is a proper fraction?

Ans: Proper Fraction

A rational fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0 is called a proper fraction if degree of the polynomial N(x) < degree of the polynomial D (x).

(iii) What is an improper fraction?

Ans: Improper Fraction

A rational fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0 is called an improper fraction if degree of the polynomial N(x) is greater or equal to the degree of D (x).

(iv) What are partial fractions?

Ans: Partial Fraction

A single fraction written in the forms of its components is said to be resolved into partial fraction.

(v) How can we make partial fractions of $\frac{x-2}{(x+2)(x+3)}$?

Ans: It is written as: $\frac{x-2}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$, then values of A and B are found as below.

$$Let \quad \frac{x-2}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \quad ....(i)$$

Multiplying both sides by $(x+2)(x+3)$, we get

$x-2 = A(x+3) + B(x+2)$ ....(ii)

Put $x+2=0 \Rightarrow x=-2$ in (ii)

$-2-2 = A(-2+3) + B(-2+2)$

$-4 = A(1)$

$\boxed{A=-4}$

Put $x+3=0 \Rightarrow x=-3$ in eq.(ii)

$-3-2 = A(-3+3) + B(-3+2)$

$-5 = B(-1)$

$\boxed{B=5}$

Putting values of A,B in (i)

$$\frac{x-2}{(x+2)(x+3)} = \frac{-4}{x+2} + \frac{5}{x+3}$$

(vi) Resolve $\frac{1}{x^2-1}$ into partial fractions.

Ans:

$$\frac{1}{x^2-1}$$
$$= \frac{1}{(x+1)(x-1)}$$

$$Let \quad \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \quad ....(i)$$

Multiplying both sides by $(x+1)(x-1)$, we get

$1 = A(x-1) + B(x+1)$ ....(ii)

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$1 = A(-1-1) + B(-1+1)$

$1 = A(-2) + B(0)$

$1 = -2A$

$\boxed{A=-\frac{1}{2}}$

Put $x-1=0 \Rightarrow x=1$ in eq.(ii)

$1 = A(1-1) + B(1+1)$

$1 = A(0) + B(2)$

$1 = 2B$

$\boxed{B=\frac{1}{2}}$

Putting values of A,B in (i)

$$\frac{1}{(x+1)(x-1)} = \frac{-1}{2(x+1)} + \frac{1}{2(x-1)}$$
$$= \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \text{(Partial Fractions)}$$

(vii) Find partial fractions of $\frac{3}{(x+1)(x-1)}$.

Ans:

$$Let \quad \frac{3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \quad ....(i)$$

Multiplying both sides by $(x+1)(x-1)$, we get

$3 = A(x-1) + B(x+1)$ ....(ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$3 = A(1-1) + B(1+1)$

$3 = 2B$

$2B = 3$

$\boxed{B=\frac{3}{2}}$

Put $x+1=0 \Rightarrow x=-1$ in eq.(ii)

$3 = A(-1-1) + B(-1+1)$

$3 = -2A$

$-2A = 3$

$\boxed{A=-\frac{3}{2}}$

Putting values of A,B in (i)

$$\frac{3}{(x+1)(x-1)} = \frac{-3}{2(x+1)} + \frac{3}{2(x-1)}$$
$$= \frac{3}{2(x-1)} - \frac{3}{2(x+1)}$$
$$= \frac{3}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

(viii) Resolve $\frac{x}{(x-3)^2}$ into partial fractions.

Ans:

$$Let \quad \frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \quad ....(i)$$

Multiplying both sides by $(x-3)^2$, we get

$x = A(x-3) + B$ ....(ii)

Put $x-3=0 \Rightarrow x=3$ in (ii)

$3 = A(3-3) + B$

$3 = B$

$B = 3$

Comparing coefficients of x

$1 = A$

$A = 1$

Putting values of A,B in (i)

$$\frac{x}{(x-3)^2} = \frac{1}{x-3} + \frac{3}{(x-3)^2}$$

(ix) How we can make the partial fractions of $\frac{x}{(x+a)(x-a)}$?

Ans:

$$Let \quad \frac{x}{(x+a)(x-a)} = \frac{A}{(x+a)} + \frac{B}{(x-a)} \quad ....(i)$$

Multiplying by $(x+a)(x-a)$, we get

$x = A(x-a) + B(x+a)$ ....(ii)

Put $x-a=0 \Rightarrow x=a$ in (ii)

$a = A(a-a) + B(a+a)$

$a = B(2a)$

$A = \frac{a}{2a}$

$\boxed{B=\frac{1}{2}}$

Put $x+a=0 \Rightarrow x=-a$ in (ii)

$-a = A(-a-a) + B(-a+a)$

$-a = A(-2a)$

$A = \frac{-a}{-2a}$

$\boxed{A=\frac{1}{2}}$

Putting values of A,B in (i)

$$\frac{x}{(x+a)(x-a)} = \frac{1}{2(x+a)} + \frac{1}{2(x-a)}$$
$$= \frac{1}{2}\left[\frac{1}{x+a} + \frac{1}{x-a}\right] \text{(Partial Fraction)}$$

(x) Whether $(x+3)^2 = x^2 + 6x + 9$ is an identity?

Ans:

An identity is an equation which is satisfied by all the values of the variables involved.

$(x+3)^2 = x^2 + 6x + 9$ ....(i)

Put $x = 7$ in (i)

$(7+3)^2 = (7)^2 + 6(7) + 9$

$100 = 100$

Put $x = -7$ in (i)

$(-7+3)^2 = (-7)^2 + 6(-7) + 9$

$(-4)^2 = 49 - 42 + 9$

$16 = 58 - 42$

$16 = 16$

Yes, this is an identity.
It is true for every value of x.

Summary

✓ A **fraction** is an indicated quotient of two numbers or algebraic expressions.

✓ An expression of the form $\frac{N(x)}{D(x)}$, with D(x) ≠ 0 and N(x) and D(x) are polynomials in x with real coefficients, is called a **rational fraction**. Every fractional expression can be expressed as a quotient of two polynomials.

✓ A rational fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0 is called a **proper fraction** if degree of the polynomial N(x) in the numerator is less than the degree of the polynomial D(x), in the denominator.

✓ A rational fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0 is called an **improper fraction** if degree of the polynomial N(x) is greater or equal to the degree of the polynomial D(x).

✓ **Partial fractions:** Decomposition of resultant fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0, when

(a) D(x) consists of non-repeated linear factors.

(b) D(x) consists of repeated linear factors.

(c) D(x) consists of non-repeated irreducible quadratic factors.

(d) D(x) consists of repeated irreducible quadratic factors.