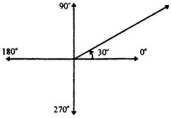
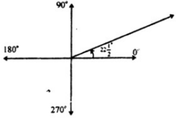
Exercise 7.1

1. Locate the following angles:

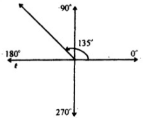
(i) 30°



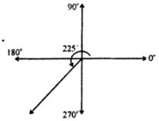
(ii) $22\frac{1}{2}^{\circ}$



(iii) 135°

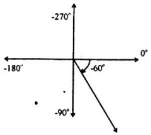


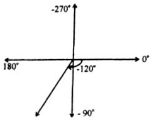
(iv) 225°



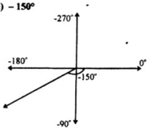
(v) – 6 0°

(vi) – 120°

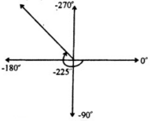




(vii) – 150°



(viii) – 225°



2. **Express the following sexagesimal measures of angles in decimal form.**

(i) 45° 30′

Solution

$$45^\circ\,30' = 45^\circ + \left(\frac{30}{60}\right)^\circ = 45^\circ + 0.5^\circ = 45.5^\circ$$

(ii) 60°30′30″

Solution:

$$\begin{aligned}60^\circ30'30'' &= 60^\circ + \left(\frac{30}{60}\right) + \left(\frac{30}{60\times60}\right)\\ &= 60^\circ + \frac{30^\circ}{60} + \frac{30^\circ}{3600}\\ &= 60^\circ + 0.5^\circ + 0.0083^\circ\end{aligned}$$

$$= 60.5083^\circ$$

(iii) 125° 22′50″

Solution

$$\begin{aligned}125^\circ\,22'50'' &= 125^\circ + \left(\frac{22}{60}\right) + \left(\frac{50}{60\times60}\right)\\ &= 125^\circ + \frac{22^\circ}{60} + \frac{50^\circ}{3600}\\ &= 125^\circ + 0.3667^\circ + 0.0139^\circ\\ &= 125.3806^\circ\end{aligned}$$

3. Express the following into D° M' S", form.

(i) 47.36⁰

Solution:

$$\begin{aligned}47.36^\circ &= 47^\circ + 0.36^\circ\\ &= 47^\circ + \left(\frac{36}{100}\right)^\circ\\ &= 47^\circ + \frac{9^\circ}{25}\\ &= 47^\circ + \left(\frac{9}{25}\times60\right)'\\ &= 47^\circ + 21.6'\\ &= 47^\circ + 21' + (0.6\times60)''\\ &= 47^\circ + 21' + 36''\\ &= 47^\circ21'36''\end{aligned}$$

(ii) 125.45⁰

Solution:

$$\begin{aligned}125.45^\circ &= 125^\circ + 0.45^\circ\\ &= 125^\circ + \left(\frac{45}{100}\right)^\circ\\ &= 125^\circ + \frac{9^\circ}{20}\\ &= 125^\circ + \left(\frac{9}{20}\times 60\right)'\\ &= 125^\circ + 27'\\ &= 125^\circ 27'\end{aligned}$$

(iii) 225.75^0

Solution:

$$\begin{aligned}225.75^\circ &= 225^\circ + 0.75^\circ\\ &= 225^\circ + \left(\frac{75}{100}\right)^\circ\\ &= 225^\circ + \frac{3^\circ}{4}\\ &= 225^\circ + \left(\frac{3}{4}\times 60\right)'\\ &= 225^\circ + 45'\\ &= 225^\circ 45'\end{aligned}$$

(iv) -22.5^0

Solution:

$$\begin{aligned}-22.5^\circ &= -22^\circ - 0.5^\circ\\ &= -22^\circ - \left(\frac{5}{10}\right)^\circ\\ &= -22^\circ - \left(\frac{5}{10}\times 60\right)'\\ &= -22^\circ - 30'\\ &= -22^\circ 30'\end{aligned}$$

(v) -67.58^0

Solution:

$$\begin{aligned}-67.58^\circ &= -67^\circ - 0.58^\circ\\ &= -67^\circ - \left(\frac{58}{100}\right)^\circ\\ &= -67^\circ - \left(\frac{58}{100}\times 60\right)'\\ &= -67^\circ - 34.8'\\ &= -67^\circ - 34' - (0.8\times 60)''\\ &= -67^\circ - 34' - 48''\\ &= -67^\circ 34' 48''\end{aligned}$$

(vi) 315.18^0

Solution:

$$\begin{aligned}315.18^\circ &= 315^\circ + 0.18^\circ\\ &= 315^\circ + \left(\frac{18}{100}\right)^\circ\\ &= 315^\circ + \left(\frac{18}{100}\times 60\right)'\\ &= 315^\circ + 10.8'\\ &= 315^\circ + 10' + (0.8\times 60)''\end{aligned}$$

$$= 315° + 10' + 48''$$
$$= 315°10'48''$$

4. Express the following angles into radians.

(i) 30°

Solution:

$$30° = 30 \times 1° = 30 \times \left(\frac{\pi}{180}\text{radians}\right)$$
$$= \frac{\pi}{6}\text{radians}$$

(ii) 60°

Solution:

$$60° = 60 \times 1° = 60 \times \left(\frac{\pi}{180}\text{radians}\right)$$
$$= \frac{\pi}{3}\text{radians}$$

(iii) 135°

Solution:

$$135° = 135 \times 1° = 135 \times \left(\frac{\pi}{180}\text{radians}\right)$$
$$= \frac{3\pi}{4}\text{radians}$$

(iv) 225°

Solution:

$$225° = 225 \times 1° = 225 \times \left(\frac{\pi}{180}\text{radians}\right)$$
$$= \frac{5\pi}{4}\text{radians}$$

(v) -150°

Solution:

$$-150° = -150 \times 1° = -150 \times \left(\frac{\pi}{180}\text{radians}\right)$$
$$= -\frac{5\pi}{6}\text{radians}$$

(vi) - 225°

Solution:

$$-225^\circ = -225\times 1^\circ = -225\times\left(\frac{\pi}{180}\text{radians}\right)$$
$$= -\frac{5\pi}{4}\text{radians}$$

(vii) 300°

Solution:

$$300^\circ = 300\times 1^\circ = 300\times\left(\frac{\pi}{180}\text{radians}\right)$$
$$= \frac{5\pi}{3}\text{radians}$$

(viii) 315°

Solution:

$$315^\circ = 315\times 1^\circ = 315\times\left(\frac{\pi}{180}\text{radians}\right)$$
$$= \frac{7\pi}{4}\text{radians}$$

5. Convert each of following to degrees.

(i) $\frac{3\pi}{4}$

Solution:

$$\frac{3\pi}{4} = \frac{3\pi}{4}\text{radians} = \frac{3\pi}{4}\times 1\text{ radian}$$
$$= \frac{3\pi}{4}\times\frac{180^\circ}{\pi} = 135^\circ$$

(ii) $\frac{5\pi}{6}$

Solution:

$$\frac{5\pi}{6} = \frac{5\pi}{6}\text{radians} = \frac{5\pi}{6}\times 1\text{ radian}$$
$$= \frac{5\pi}{6}\times\frac{180^\circ}{\pi} = 150^\circ$$

(iii) $\frac{7\pi}{8}$

Solution:

$$\frac{7\pi}{8} = \frac{7\pi}{8}\text{radians} = \frac{7\pi}{8}\times 1\text{ radian}$$
$$= \frac{7\pi}{8}\times\frac{180^\circ}{\pi} = 157.5^\circ$$

(iv) $\frac{13\pi}{16}$

Solution:

$$\frac{13\pi}{16} = \frac{13\pi}{16}\text{radians} = \frac{13\pi}{16}\times 1\text{ radian}$$
$$= \frac{13\pi}{16}\times\frac{180^\circ}{\pi} = 146.25^\circ$$

(v) 3

Solution:

$$\begin{aligned} 3 &= 3 \text{ radians} \quad = 3\times 1 \text{ radian} \\ &= 3\times\frac{180^\circ}{\pi} \quad = 3\times 57.295779^\circ \\ &= 171.89^\circ \end{aligned}$$

(vi) 4.5

Solution:

$$\begin{aligned} 4.5 &= 4.5 \text{ radians} \quad = 4.5\times 1 \text{ radian} \\ &= 4.5\times\frac{180^\circ}{\pi} \quad = 4.5\times 57.295779^\circ \\ &= 257.83^\circ \end{aligned}$$

(vii) $\frac{-7\pi}{8}$

Solution:

$$\begin{aligned} \frac{-7\pi}{8} = \frac{-7\pi}{8}\text{radians} &= \frac{-7\pi}{8}\times 1 \text{ radian} \\ &= -\frac{7\pi}{8}\times\frac{180^\circ}{\pi} = -157.5^\circ \end{aligned}$$

(viii) $-\frac{13\pi}{8}$

Solution:

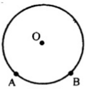
$$\begin{aligned} -\frac{13\pi}{8} = -\frac{13\pi}{8}\text{radians} &= -\frac{13\pi}{8}\times 1 \text{ radians} \\ &= -\frac{13\pi}{8}\times\frac{180^\circ}{\pi} = -146.25^\circ \end{aligned}$$

Sector of a Circle

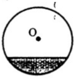
(i) A part of the circumference of a circle is called an arc.

(ii) A part of the circle bounded by an arc and a chord is called segment of a circle.

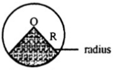
(iii) A part of the circle bounded by the two radii and an arc is called sector of the circle.



an arc

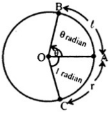


a segment



a sector

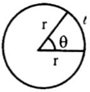
To establish the rule l = rθ, where r is the radius of the circle, l the length of circular arc and θ the central angle measured in radians:



Let an arc AB denoted by l subtends an angle θ radian at the center of the circle. It is a fact of plane geometry that measure of central angles of the arcs of a circle are proportional to the lengths of their arcs.

$$\frac{m\angle \mathrm{AOB}}{m\angle \mathrm{AOC}} = \frac{m\overset{\frown}{AB}}{m\overset{\frown}{AC}}$$

$$\Rightarrow \frac{\theta\ \text{radian}}{1\ \text{radian}} = \frac{l}{r} \Rightarrow \frac{l}{r} = \theta \text{ or } l = r\theta$$



Area of a circular sector

Consider a circle of radius r units and an arc of length l units, subtending an angle θ at O.

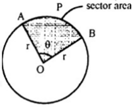


Fig. 7.2.2

Area of the circle = πr^2

Angle of the circle = 2π

Angle of the sector = θ radian

Then by elementary geometry, we can use the proportion,

$$\frac{\text{area of sector AOBP}}{\text{area of circle}} = \frac{\text{angle of the sector}}{\text{angleof the circle}}$$

$$or \ \frac{\text{area of sector AOBP}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \text{area of sector AOBP} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta$$

EXERCISE 7.2

1. Find θ, when:

(i) l = 4.5m , r = 3.5m

Solution:

We know that

$$l = r\theta \qquad \Rightarrow \theta = \frac{l}{r}$$

$$\theta = \frac{2}{3.5} \qquad \Rightarrow \theta = 0.57 \text{ radians}$$

(ii) l = 4.5m , r = 2.5m

Solution:

We know that

$$l = r\theta \qquad \Rightarrow \theta = \frac{l}{r}$$

$$\theta = \frac{4.5}{2.5} \qquad \Rightarrow \theta = 1.8 \text{ radians}$$

2. Find l, when

(i) θ =180° , r = 4.9 cm

Solution:

$$\theta = 180^\circ = 180^\circ \times 1^\circ = 180^\circ \times \frac{\pi}{180^\circ} = \ 3.1415 \text{ radians}$$

We know that

$$l = r\theta \qquad \Rightarrow \theta = (4.9)(3.1415)$$

$$\theta = 15.4cm$$

(ii) θ =60° 30' , r = 15mm

Solution:

$$\theta = 60^\circ 30' = 60^\circ + \left(\frac{30}{60}\right)^\circ = 60^\circ + 0.5 = 60.5^\circ$$

$$= 60.5 \times 1^\circ = 60.5^\circ \times \frac{\pi}{180} = 1.056 \text{ radians}$$

We know that

$$l = r\theta \qquad \Rightarrow \theta = (15)(1.056)$$

$$\theta = 15.84mm$$

3. Find r, when:

(i) l = 4cm , θ = $\frac{1}{4}$ radian

Solution:

We know that

$$l = r\theta \qquad \Rightarrow r = \frac{l}{\theta}$$

$$r = \frac{4}{1/4} \qquad \Rightarrow r = 4 \times \frac{4}{1} = 16cm$$

(ii) l = 52cm , θ = 45°

Solution:

We know that

$$\theta = 45^\circ = 45 \times 1^\circ = 45 \times \frac{\pi}{180} = \frac{\pi}{4} \text{ radians}$$

$$= 0.7854 \text{ radians}$$

We know that

$$l = r\theta \qquad \Rightarrow r = \frac{l}{\theta}$$

$$r = \frac{52}{0.7854} \qquad \Rightarrow r = 66.21cm$$

4. In a circle of radius 12m, find the length of an arc which subtends a central angle θ = 1.5 radian.

Solution:

r= 12 cm, θ = 1.5 radians, l =**?**

We know that

l = rθ l = (12) (1.5) ⇒ l = 18 m

5. In a circle of radius 10m, find the distance travelled by a point moving on this circle if the point makes 3.5 revolution. (3.5 revolution = 7π).

Solution:

r = l0m, θ = 3.5 revolution = 7 π = 7 x 3.1415 = 22 rad.

We know that

l = rθ

l = (10) (22) l = 220 m

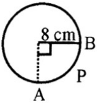
6. What is the circular measure of the angle between the hands of the watch at 3 o'clock?

Solution:

The angle between the hands of a watch at 3o' clock is

$$= \left(\frac{360}{12}\times 3\right)^{\circ} = (30\times 3)^{\circ} = 90^{\circ}$$
$$= 90\times 1^{\circ}$$
$$= 90\times\frac{\pi}{180} = \frac{\pi}{2}\text{ radians}$$

7. What is the length of the arc APB?



Solution:

$$Arc\ \overset{\frown}{ABP} = ?;\ \ r = 8cm$$
$$\theta = 90^{\circ} = 90\times 1^{\circ} = 90\times\frac{\pi}{180}\text{radians}$$
$$= \frac{\pi}{2}\text{radians} = 1.57\text{radians}$$

We know that

$l = r\theta$

$l = (8)(1.57) = 12.57$ cm

8. In a circle of radius 12cm, how long an arc subtends a central angle of 84°.

Solution:

$r = 12cm \qquad l = ?$

$$\theta = 84^{\circ} = 84\times 1^{\circ} = 84\times\frac{\pi}{180}\text{radians}$$
$$= 1.4661\text{ radians}$$

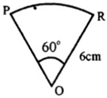
We know that

$l = r\theta$

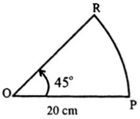
$l = (12)(1.4661) = 17.6$ cm

9. Find the area of the sectors OPR.

(a)



(b)



Solution:

$(a)\,Here, r = 6cm, \theta = 60^\circ = 60\times 1^\circ = 60\times\frac{\pi}{180} = \frac{\pi}{3}\text{rad.}$

We know that

$$\begin{aligned}\text{Area of the sector } &= \frac{1}{2}r^2\theta\\ &= \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right)\\ &= \frac{1}{2}36\left(\frac{\pi}{3}\right)\\ &= 6\pi\\ &= 6\times\frac{22}{7}\\ &= 18.86cm^2\end{aligned}$$

$(b)\,Here\ r = 20cm, \theta = 45^\circ = 45\times 1^\circ = 45\times\frac{\pi}{180} = \frac{\pi}{4}\text{rad.}$

We know that

$$\begin{aligned}\text{Area of the sector } &= \frac{1}{2}r^2\theta\\ &= \frac{1}{2}(20)^2\left(\frac{\pi}{4}\right)\\ &= \frac{1}{2}(400)\left(\frac{\pi}{4}\right)\\ &= 50\pi\\ &= 50\times\frac{22}{7}\end{aligned}$$

$$= 157.14cm^2$$

10. Find area of the sector inside a central angle of 20° in a circle of radius 7m.

Solution:

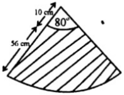
$Area = ?\ r = 7m, \theta = 20^\circ = 20\times 1^\circ = 20\times\frac{\pi}{180}\text{rad.}$

We know that

$$\begin{aligned}\text{Area of the sector } &= \frac{1}{2}r^2\theta\\ &= \frac{1}{2}(49)\left(\frac{\pi}{9}\right)\\ &= \frac{49\pi}{18}m^2\\ &= \frac{49\times 22}{18\times 7}\\ &= \frac{77}{9}\\ &= 8.55m^2\end{aligned}$$

11. Sehar is making a skirt. Each panel of this skirt is of the shape shown

shaded in the diagram. How much material (cloth) is required for each panel?



Solution:

$r = 56cm + 10cm = 66cm$

$\theta = 80^\circ = 80\times 1^\circ = 80\times\frac{\pi}{180} = \frac{4}{9}\pi = \frac{4}{9}\times\frac{22}{7}\text{ rad.} = \frac{88}{63}\text{ radians}$

We know that

$$\begin{aligned}\text{Area of the sector } &= \frac{1}{2}\text{r}^2\theta \\ &= \frac{1}{2}(66)^2\left(\frac{88}{63}\right) \\ &= \frac{66\times 66\times 88}{2\times 63} \\ &= 2972.39cm^2\end{aligned}$$

12. Find the area of the sector with central angle of $\frac{\pi}{5}$ radian in a circle of radius 10cm.

Solution:

$\theta = \frac{\pi}{5}rad.$ $\quad r = 10cm,$ $\quad lrea = ?$

We know that

$$\begin{aligned}\text{Area of the sector } &= \frac{1}{2}\text{r}^2\theta \\ &= \frac{1}{2}(10)^2\left(\frac{\pi}{5}\right) \\ &= \frac{1}{2}(100)\left(\frac{\pi}{5}\right) \\ &= 10\pi\ cm^2 \\ &= 10\times\frac{22}{7} \\ &= 31.43cm^2\end{aligned}$$

13. The area of the sector with a central angle θ in a circle of radius 2m is 10 square meters. Find 8 in radians.

Solution:

$r = 2m, \quad Area = 10m^2 \qquad \theta = ?$

We know that

Area of the sector $= \frac{1}{2} r^2 \theta$

$$10 = \frac{1}{2}(2)^2 \theta$$

$$10 = 2\theta$$

$$\theta = \frac{10}{2}$$

$\theta = 5$ radians

Trigonometric Ratios:

Coterminal Angle

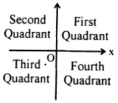
Two or more than two angles with the same initial and terminal sides are called coterminal angles.

It means that terminal side comes to its original position after every revolution of 2π radian in anti-clockwise or clockwise direction. In general if θ is in degrees, then $360°k + \theta$, where $k \in Z$, is an angle coterminal with θ. If angle θ is in radian measure, then $2k\pi + \theta$ where $k \in Z$, is an angle coterminal with θ.

Thus, the general angle $\theta = 2(k)\,\pi + \theta$, where $k \in Z$.

The Quadrants and Quadrantal Angles:

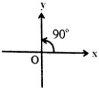
The x - axis and y-axis divides the plane in four regions, called quadrants, when they intersect each other at right angle. The point of intersection is called origin and is denoted by O.

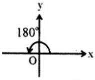


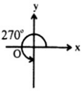
Angles between 0° and 90° are in the first quadrant. Angles between 90° and 180° are in the second quadrant. Angles between 180° and 270° are in the third quadrant. Angles between 270° to 360° are in the fourth quadrant. An angle in standard position is said to lie in a quadrant if its terminal side lies in that quadrant.

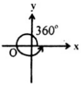
Quadrantal Angles:

If the terminal side of an angle in standard position falls on x-axis or y-axis, then it is called a quadrantal angle i.e., 90°, 180°, 270° and 360° are quadrantal angles. The quadrantal angles are shown as below:





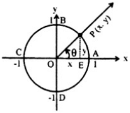




Trigonometric ratios and their reciprocals with the help of a unit circle:

There are six fundamental trigonometric ratios called sine, cosine, tangent, cotangent, secant and cosecant. To define these functions, we use circular approach which involves the unit circle.

Let θ be a real number, which represents the radian measure of an angle in standard position. Let P (x, y) be any point on the unit circle lying on terminal side of θ as shown in the figure.



We define sine of θ, written as sinθ and cosine of θ written as cosθ as:

$$\sin\theta = \frac{EP}{OP} = \frac{y}{1} \quad \Rightarrow \quad \sin\theta = y$$

$$\text{and} \quad \cos\theta = \frac{OE}{OP} = \frac{x}{1} \quad \Rightarrow \quad \cos\theta = x$$

i.e., cosθ and sinθ are the x-coordinate and y-coordinate of the point P on the unit circle. The equations x = cos θ and y = sin θ are called circular or trigonometric functions.

The remaining trigonometric functions tangent, cotangent, secant and cosecant will be denoted by tan θ, cotθ, secθ and cosec θ for any real angle θ.

$$\tan\theta = \frac{EP}{OE} = \frac{y}{x} \quad \Rightarrow \quad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\text{since} \quad y = \sin\theta \text{ and } x = \cos\theta \quad \Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} \ (y \neq 0) \quad \Rightarrow \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{x} \ (x \neq 0) \quad \text{and} \quad \operatorname{cosec}\theta = \frac{1}{y} \ (y \neq 0)$$

$$= \frac{1}{\cos\theta} \qquad\qquad = \frac{1}{\sin\theta}$$

Reciprocal Identities

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \quad \text{or} \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{or} \quad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{or} \quad \cot\theta = \frac{1}{\tan\theta}$$

The values of trigonometric ratio for 45°, 30°, 60°:

Consider a right triangle ABC with m∠C = 90°. The sides opposite to the vertices A, B and C are denoted by a, b and c respectively.

Case I

When m∠A = 45°, where 45° = $\frac{\pi}{4}$ radian. Since the sum of angles in a triangle is 180°, so m∠B = 45°.

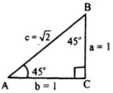


Fig. 7.3.5

As values of trigonometric functions depends on the size of the angle only and not on the size of triangle. For convenience, we take a = b = 1. In this case the triangle is isosceles right triangle.

By Pythagorean theorem, we have

$c^2 = a^2 + b^2 \Rightarrow c^2 = (l)^2 + (l)^2 = 2$

$c^2 = 2 \Rightarrow c = \sqrt{2}$

From this triangle we have

$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{a}{c} = \frac{1}{\sqrt{2}}; \qquad \operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$$

$$\cos 45^\circ = \cos\frac{\pi}{4} = \frac{b}{c} = \frac{1}{\sqrt{2}}; \qquad \sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2}$$

$$\tan 45^\circ = \tan\frac{\pi}{4} = \frac{a}{b} = \frac{1}{1} = 1; \qquad \cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

Case II

When m ∠ A = 30° or m ∠ A = 60°

Consider an equilateral triangle with sides a = b = c = 2 for convenience. Since the angles in an equilateral triangle are equal and their sum is 180°, each angle has measure 60°.

Bisecting an angle in the triangle, we obtain two right triangles with 30° and 60° angles. The height |AD| of these triangles may be found by Pythagorean theorem, i.e.,

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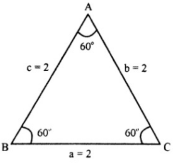
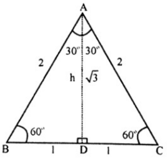


(i)



(ii)

$$(AD)^2 + (BD)^2 = (AB)^2 - \Rightarrow (AD)^2 = (AB)^2 - (BD)^2$$

$$h^2 = (2)^2 - (1)^2 = 3$$

$$\Rightarrow \quad h = \sqrt{3}$$

Using triangle ADB with m ∠ A = 30°, we have

$$\sin 30^\circ = \sin\frac{\pi}{6} = \frac{BD}{AB} = \frac{1}{2}; \qquad \operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\cos 30^\circ = \cos\frac{\pi}{6} = \frac{AD}{AB} = \frac{\sqrt{3}}{2}; \qquad \sec 30^\circ = \frac{1}{\cos 30^\circ} = 2$$

$$\tan 30^\circ = \tan\frac{\pi}{6} = \frac{BD}{AD} = \frac{1}{\sqrt{3}}; \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

Now Using triangle ABD with m ∠ B = 60°, we have

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}; \qquad \operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{1}{2}; \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}}{1}; \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$$

Signs of trigonometric ratios in different quadrants:

In case of trigonometric ratios like sinθ, cosθ and tanθ if θ is not a quadrantal angle, then θ will lie in a particular quadrant. Since $r = \sqrt{x^2 + y^2}$ is always + ve, the signs of ratios can be found if the quadrant of θ is known.

(i) If θ lies in first quadrant then a point P (x, y) on its terminal side has x and y coordinate positive.

Therefore, all trigonometric functions are positive in quadrant I.

(ii) If θ lies in 2nd quadrant, then point P (x, y) on its terminal side has negative x-coordinate and positive y-coordinate.

$\therefore \sin\theta = \frac{y}{r}$ is + ve or > 0 $\quad \cos\theta = \frac{x}{r}$ is -ve or < 0 and $\tan\theta = \frac{y}{x}$ is -ve or < 0

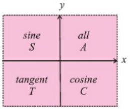
(iii) When θ lies in third quadrant, then a point P (x, y) on its terminal side has negative x-coordinate and negative y-coordinate.

$\therefore \sin\theta = \frac{y}{r}$ is -ve or < 0 $\quad \cos\theta = \frac{x}{r}$ is -ve or < 0 and $\tan\theta = \frac{y}{x}$ is +ve or > 0

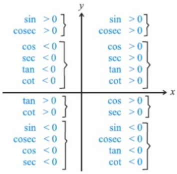
(iv) When θ lies in fourth quadrant, then the point P (x, y) on the terminal side of θ has positive x-coordinate and negative y-coordinate.

$\therefore \sin\theta = \frac{y}{r}$ is -ve or < 0 $\cos\theta = \frac{x}{r}$ is + ve or > 0 and $\tan\theta = \frac{y}{x}$ is -ve or < 0

The signs of all trigonometric functions are summarized as below.



Read 'ASTC' as
'Add Sugar To Coffee'



Calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°:

An angle θ is called a quadrantal angle if its terminal side lies on the x-axis or on the y-axis.

Case I

When θ = 0°

The point (1, 0) lies on the terminal side of angle θ°. We may consider the point on the unit circle on the terminal side of the angle.

P(1,0) x = I and y = 0 so $r=\sqrt{x^2+y^2}=\sqrt{1+0}=1$

$$\sin 0^\circ=\frac{y}{r}=\frac{0}{1}=1; \qquad \operatorname{cosec} 0^\circ=\frac{1}{\sin 0^\circ}=\frac{1}{0}=\infty(\text{undefined})$$

$$\cos 0^\circ=\frac{x}{r}=\frac{1}{1}=1; \qquad \sec 0^\circ=\frac{1}{\cos 0^\circ}=1$$

$$\tan 0^\circ=\frac{y}{r}=\frac{0}{1}=1; \qquad \cot 0^\circ=\frac{1}{\tan 0^\circ}=\frac{1}{0}=\infty(\text{undefined})$$

Case II When θ = 90°

The point P (0, 1) lies on the terminal side of angle 90°

Here x = 0 and y = 1 ⇒ $r=\sqrt{0^2+(1)^2}=1$

---

sin 90° = $\frac{y}{r}=\frac{1}{1}=1$

i.e., sin 90° = 1 and cosec 90° = $\frac{r}{y}=1$

Using reciprocal identities, we have

cos 90° = $\frac{x}{r}=\frac{0}{1}=0$; sec 90° = $\frac{r}{x}=\frac{1}{0}=\infty$ (undefined)

tan 90° = $\frac{y}{x}=\frac{1}{0}=\infty(\text{undefined})$, cot 90° = $\frac{x}{y}=\frac{0}{1}=0$

Case III When θ° = 180°

The point P (-1, 0) lies on x'-axis or on terminal side of angle 180°

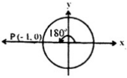
Here x = - 1 and y = 0

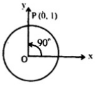
$\Rightarrow=\sqrt{x^2+y^2}=1$

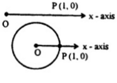
$$\therefore \sin 180^\circ=\frac{y}{r}=\frac{0}{1}=0: \qquad \operatorname{cosec} 180^\circ=\frac{r}{y}=\frac{1}{0}=\infty(\text{undefined})$$

$$\cos 180^\circ=\frac{x}{r}=\frac{-1}{1}=-1; \qquad \sec 180^\circ=\frac{r}{x}=\frac{1}{-1}=-1$$

$$\tan 180^\circ=\frac{y}{x}=\frac{0}{-1}=0; \qquad \cot 180^\circ=\frac{x}{y}=\frac{-1}{0}=\infty(\text{undefined})$$







Case IV

When θ = 270° and the point P (0, -1) lies on y-axis or on the terminal side of angle 270°.

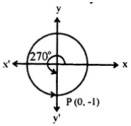
The point P (0, -1) shows that x = 0 and y = -1.

So $r = \sqrt{(0)^2 + (-1)^2} = 1$

$$\therefore \sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1; \qquad \operatorname{cosec} 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0; \qquad \sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \infty$$

$$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = -\infty; \qquad \cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$$



Case V

When $\theta°$ = 360°

Now the point P (1, 0) lies once again on x – axis.

We know that $2k\pi + \theta = \theta$ where $k \in Z$

Now $\theta = 360^\circ = 0^\circ + (360^\circ)1 = 0^\circ$ where $k = 1$

So

$$\sin 360^\circ = \sin 0^\circ = 0; \qquad \operatorname{cosec} 360^\circ = \frac{1}{\sin 360^\circ} = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty \text{(undefined)}$$

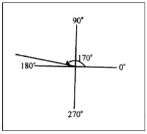
$$\cos 360^\circ = \cos 0^\circ = 1; \qquad \sec 360^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\tan 360^\circ = \tan 0^\circ = 0; \qquad \cot 360^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = \infty \text{(undefined)}$$

Exercise 7.3

1. Locate each of the following angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle conterminal with each given angle.

(i) 170°



Solution:

Positive Conterminal angle

= 170° + 360°

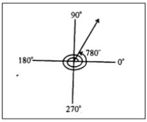
= 530°

Negative Coterminal angle

= 170° - 360°

= -190°

(ii) 780°



Solution:

Positive conterminal angle

780° = 60° + 2(360°)

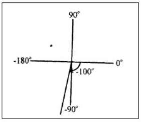
=60°

Negative Coterminal angle

= 60° - 360°

= -300°

(iii) -100°



Solution:

Positive conterminal angle

= -100° + 360°

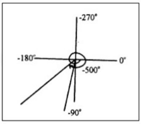
= 260°

Negative Conterminal angle

= -100° - 360°

= -460°

(iv) -500°



Solution:

Positive conterminal angle

= -500° + 2(360°,

= -500° + 720°

= 220°

Negative Conterminal angle

= 360° - 500°

= -140°

2. Identify the closest quadrantal angles between which the following angles lies.

(i) 156°

Solution:

The closest quadrantal angles between which 156° lies are 90° and 180°.

(ii) 318°

Solution:

The closest quadrantal angles between which 318° lies are 270° and 360°.

(iii) 572°

Solution

The closest quadrantal angles between which 572° lies are 540° and 630°.

(iv) -330°

Solution:

The closest quadrantal angles between which -330° lies are 0° and 90°.

3. Write the closest quadrantal angles between which the angle lies. Write your answer in radian measure.

(i) $\frac{\pi}{3}$

Solution:

The closest quadrantal angles between which $\frac{\pi}{3}$ lies are 0 and $\frac{\pi}{2}$.

(ii) $\frac{3\pi}{4}$

Solution:

The closest quadrantal angles between which $\frac{3\pi}{4}$ lies are $\frac{\pi}{2}$ and π.

(iii) $\frac{-\pi}{4}$

Solution

The closest quadrantal angles between which $\frac{-\pi}{4}$ lies are 0 and $-\frac{\pi}{2}$.

(iv) $\frac{-3\pi}{4}$

Solution

The closest quadrantal angles between which $\frac{-3\pi}{4}$ lies are $-\frac{\pi}{2}$ and $-\pi$.

4. In which quadrant (9 lie when

(i) $\sin\theta > 0$, [redacted]

Solution

= Quadrant II

(ii) $\cos\theta < 0$, $\sin\theta < 0$

Solution

= Quadrant III

(iii) $\sec\theta > 0$, $\sin\theta < 0$

Solution

= Quadrant IV

(iv) $\cos\theta < 0$, $\tan\theta < 0$

Solution

= Quadrant II

(v) $\mathrm{cosec}\,\theta > 0$, $\cos\theta > 0$

Solution

= Quadrant I

(vi) $\sin\theta < 0$, $\sec\theta < 0$

Solution

= Quadrant III

5. Fill in the blanks.

(i) cos (-150°) = ……. cos150°

Solution:

= +ve

(ii) sin (- 310°) = ……. sin310°

Solution:

= -ve

(iii) tan (- 210°) = ……. tan 210°

Solution:

= -ve

(iv) cot (- 45°) = ……. cot 45°

Solution:

= -ve

(v) sec (- 60°) = ……. sec 60°

Solution:

= +ve

(vi) cosec (- 137°) = ……. cosec 137°

Solution:

= -ve

6. The given point P lies on the-terminal side of 0. Find quadrant of 6 and all six trigonometric ratios.

(i) (-2, 3)

Solution

(-2, 3)

Here x = - 2 and y = 3

So, the quadrant of θ is II.

Now, by Pythagoras theorem,

We have

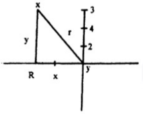
$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{(-2)^2 + (3)^2} \\
&= \sqrt{4+9} \\
&= \sqrt{13}
\end{aligned}
$$

The six trigonometric ratios are

$$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} \qquad \operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{13}}{3}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{13}}{-2}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{-2} \qquad \cot\theta = \frac{x}{y} = \frac{-2}{3}$$



(ii) (-3, -4)

Solution

(-3, -4)

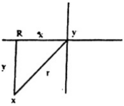
Here x = - 3 and y = -4

So, the quadrant of θ is III.

Now, by Pythagoras theorem,

We have

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{(-3)^2 + (-4)^2} \\
&= \sqrt{9+16} \\
&= \sqrt{25} = 5
\end{aligned}
$$



The six trigonometric ratios are

$$\sin\theta = \frac{y}{r} = \frac{-4}{5} \qquad \operatorname{cosec}\theta = \frac{r}{y} = \frac{5}{-4}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5} \qquad \sec\theta = \frac{r}{x} = \frac{5}{-3}$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{-4} = \frac{3}{4}$$

(iii) ($\sqrt{2}$,1)

Solution

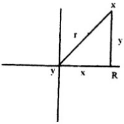
($\sqrt{2}$, 1)

Here x = $\sqrt{2}$ and y = 1.

So, the quadrant of θ is I.

Now, by Pythagoras theorem,

We have



$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{\left(\sqrt{2}\right)^2+(1)^2} \\ &= \sqrt{2+1} \\ &= \sqrt{3} \end{aligned}$$

The six trigonometric ratios are

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{3}}$ $\qquad$ $\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$

$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{2}}$ $\qquad$ $\cot\theta = \frac{x}{y} = \frac{\sqrt{2}}{1} = \sqrt{2}$

7. If cos θ = $\frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, find the values of remaining trigonometric functions.

Solution:

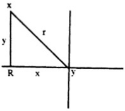
As cos θ = $\frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, so

x = - 2 and r = 3

Now, by Pythagoras theorem,

We have

$$\begin{aligned} r^2 &= x^2 + y^2 \\ y^2 &= r^2 - x^2 \\ y &= \sqrt{r^2-x^2} \\ y &= \sqrt{(3)^2-(-2)^2} \\ y &= \sqrt{9-4} \\ y &= \sqrt{5} \end{aligned}$$



The six trigonometric ratios are

$\sin\theta = \frac{y}{r} = \frac{\sqrt{5}}{3}$ $\qquad$ $\operatorname{cosec}\theta = \frac{r}{y} = \frac{3}{\sqrt{5}}$

$$\cos\theta = \frac{x}{r} = \frac{-2}{3} \qquad \sec\theta = \frac{r}{x} = \frac{3}{-2}$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{5}}{-2} \qquad \cot\theta = \frac{x}{y} = \frac{-2}{\sqrt{5}}$$

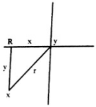
8. If tanθ = $\frac{4}{3}$ and sinθ < 0, find the values of other trigonometric functions at θ.

Solution:

As tan θ = $\frac{4}{3}$ and sinθ < 0 (terminal arm of the angle θ is in quadrant III), so,

x = 3 and y = 4

Now, by Pythagoras theorem, we have



$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(3)^2 + (4)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} = 5 \end{aligned}$$

The six trigonometric ratios are

$$\sin\theta = -\frac{y}{r} = -\frac{4}{5} \qquad \operatorname{cosec}\theta = -\frac{r}{y} = -\frac{5}{4}$$

$$\cos\theta = -\frac{x}{r} = -\frac{3}{5} \qquad \sec\theta = -\frac{r}{x} = -\frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{3} \qquad \cot\theta = \frac{x}{y} = \frac{3}{4}$$

9. If sinθ = $-\frac{1}{\sqrt{2}}$ and terminal side of the angle is not in quadrant II, find the values of tanθ, secθ, and cosecθ.

As sinθ = $-\frac{1}{\sqrt{2}}$ and terminal side of the angle is in quadrant - IV, so,

y = -1 and r = $\sqrt{2}$

Now, by Pythagoras theorem, we have

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x^2 &= r^2 - y^2 \\ x &= \sqrt{r^2 - y^2} \\ x &= \sqrt{\left(\sqrt{2}\right)^2 - (-1)^2} \\ x &= \sqrt{2-1} \\ x &= \sqrt{1} \\ x &= 1 \end{aligned}$$

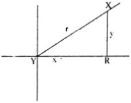
Now,

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$

10. If cosecθ = $\frac{13}{12}$ and sec θ > 0, find the remaining trigonometric functions.

As cosecθ = $\frac{13}{12}$ and Secϑ > 0 (terminal arm of angle θ is in quadrant I), so y = 12 and r = 13



By Pythagoras theorem, we have

$$r^2 = x^2 + y^2$$
$$x^2 = r^2 - y^2$$
$$x = \sqrt{r^2 - y^2}$$
$$x = \sqrt{(13)^2 - (12)^2}$$
$$x = \sqrt{169 - 144}$$
$$x = \sqrt{25}$$
$$x = 5$$

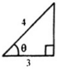
The remaining trigonometric ratios are

$$\sin\theta = \frac{y}{r} = \frac{12}{13}$$

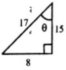
$$\cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \sec\theta = \frac{r}{x} = \frac{13}{5}$$
$$\tan\theta = \frac{y}{x} = \frac{12}{5} \qquad \cot\theta = \frac{x}{y} = \frac{5}{12}$$

11. Find the values of trigonometric functions at the indicated angle θ in the right triangle.

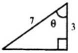
(i)



(ii)



(iii)



(i) As x = 3 and r = 4

By Pythagoras theorem, we have

$$r^2 = x^2 + y^2$$
$$v^2 = r^2 - x^2$$
$$y = \sqrt{r^2 - x^2}$$
$$y = \sqrt{(4)^2 - (3)^2}$$
$$y = \sqrt{16 - 9}$$
$$y = \sqrt{7}$$

The six trigonometric ratios are

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{7}}{4} \qquad \operatorname{cosec}\theta = \frac{r}{y} = \frac{4}{\sqrt{7}}$$
$$\cos\theta = \frac{x}{r} = \frac{3}{4} \qquad \sec\theta = \frac{r}{x} = \frac{4}{3}$$
$$\tan\theta = \frac{y}{x} = \frac{\sqrt{7}}{3} \qquad \cot\theta = \frac{x}{y} = \frac{3}{\sqrt{7}}$$

(ii) As r = 17, x = 15, y = 8

The six trigonometric ratios are

$\sin\theta=\frac{y}{r}=\frac{8}{17}$ $\qquad$ $\operatorname{cosec}\theta=\frac{r}{y}=\frac{17}{8}$

$\cos\theta=\frac{x}{r}=\frac{15}{17}$ $\qquad$ $\sec\theta=\frac{r}{x}=\frac{17}{15}$

$\tan\theta=\frac{y}{x}=\frac{8}{15}$ $\qquad$ $\cot\theta=\frac{x}{y}=\frac{15}{8}$

(iii) As x = 3 and r = 7

By Pythagoras theorem, we have

$$r^2=x^2+y^2$$
$$y^2=r^2-x^2$$
$$y=\sqrt{r^2-x^2}$$
$$y=\sqrt{(7)^2-(3)^2}$$
$$y=\sqrt{49-9}$$
$$y=\sqrt{40}$$
$$y=2\sqrt{10}$$

The six trigonometric ratios are

$\sin\theta=\frac{x}{r}=\frac{2\sqrt{10}}{7}$ $\qquad$ $\operatorname{cosec}\theta=\frac{r}{y}=\frac{7}{2\sqrt{10}}$

$\cos\theta=\frac{x}{r}=\frac{3}{7}$ $\qquad$ $\sec\theta=\frac{r}{x}=\frac{7}{3}$

$\tan\theta=\frac{y}{x}=\frac{2\sqrt{10}}{3}$ $\qquad$ $\cot\theta=\frac{x}{y}=\frac{3}{2\sqrt{10}}$

12. Find the values of the trigonometric functions. Do not use trigonometric tables or calculator.

(i) tan30°

Solution:

$$\tan 30^\circ=\frac{1}{\sqrt{3}}$$

(ii) tan330°

Solution:

We know that

$2k\pi+\theta=\theta$ where $k\in z$

$$\begin{aligned}\tan 330^\circ&=\tan\left(360^\circ-30^\circ\right)\\&=\tan\left(2(1)\pi-30^\circ\right)\\&=\tan\left(-30^\circ\right)\\&=-\tan 30^\circ\\&=-\frac{1}{\sqrt{3}}\end{aligned}$$

(iii) sec330°

Solution:

We know that

$2k\pi+\theta=\theta$ where $k\in z$

$$\begin{aligned}\sec 330^\circ&=\sec\left(360^\circ-30^\circ\right)\\&=\sec\left(2(1)\pi-30^\circ\right)\\&=\sec\left(-30^\circ\right)\end{aligned}$$

$$= \sec 30^{\circ}$$

$$= \frac{1}{\cos 30^{\circ}}$$

$$= \frac{2}{\sqrt{3}}$$

(iv) $\cot\frac{\pi}{4}$

Solution:

$$\cot\frac{\pi}{4} = \frac{1}{\tan\frac{\pi}{4}}$$

$$= \frac{1}{1} = 1$$

(v) $\cos\frac{2\pi}{3}$

Solution:

We know that

$2k\pi + \theta, \theta$ where $k \in z$

$$\cos\left(\frac{2\pi}{3}\right) = \cos\left(2\left(\frac{1}{2}\right)\pi - \frac{\pi}{3}\right)$$

$$= \cos\left(-\frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{3}$$

$$= -\frac{1}{2} \because (\text{In quad,II } \cos\theta < 0)$$

(vi) $\operatorname{cosec}\frac{2\pi}{3}$

Solution:

We know that

$2k\pi + \theta = \theta$

$$\operatorname{cosec}\left(\frac{2\pi}{3}\right) = \operatorname{cosec}\left(2\left(\frac{1}{2}\right) - \frac{\pi}{3}\right)$$

$$= \operatorname{cosec}\left(-\frac{\pi}{3}\right)$$

$$= \frac{1}{\sin\left(-\frac{\pi}{3}\right)}$$

$$= -\frac{1}{\sin\left(\frac{\pi}{3}\right)}$$

$$
\begin{aligned}
&= \frac{1}{\sin\left(\frac{\pi}{3}\right)} \because (\text{In quad,II } \sin\theta > 0)\\
&= \frac{1}{\sqrt{3/2}}\\
&= \frac{2}{\sqrt{3}}
\end{aligned}
$$

(vii) cos (-450°)

Solution:

We know that

$2k\pi + \theta = \theta$

$$
\begin{aligned}
\cos(-450^\circ) &= \cos(2(-1)\pi - 90^\circ)\\
&= \cos(-90^\circ)\\
&= \cos 90^\circ = 0
\end{aligned}
$$

(viii) tan (-9π)

Solution:

We know that

$2k\pi + \theta = \theta$

$$
\begin{aligned}
\tan(-9\pi) &= \tan(2(-5)\pi + \pi)\\
&= \tan(\pi)\\
&= 0
\end{aligned}
$$

(ix) cos $\left(\frac{-5\pi}{6}\right)$

Solution:

We know that

$2k\pi + \theta = \theta$

$$
\begin{aligned}
\cos\left(\frac{-5\pi}{6}\right) &= \cos\left(2\left(-\frac{1}{2}\right)\pi + \frac{\pi}{6}\right)\\
&= \cos\left(\frac{\pi}{6}\right)\\
&= -\frac{\sqrt{3}}{2}
\end{aligned}
$$

$\because (\text{In quad,II } \cos\theta < 0)$

(x) sin $\frac{7\pi}{6}$

Solution:

We know that

$2k\pi + \theta = \theta$

$$
\begin{aligned}
\sin\left(\frac{7\pi}{6}\right) &= \sin\left(2\left(\frac{1}{2}\right)\pi + \frac{\pi}{6}\right)\\
&= \sin\left(\frac{\pi}{6}\right)\\
&= -\frac{1}{2} \because (\text{In quad,III } \sin\theta < 0)
\end{aligned}
$$

(xi) $\cot\left(\frac{7\pi}{6}\right)$

Solution:

We know that

$$2k\pi + \theta = \theta$$

$$\sin\left(\frac{7\pi}{6}\right) = \sin\left(2\left(\frac{1}{2}\right)\pi + \frac{\pi}{6}\right)$$

$$= \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{\tan(\pi/6)}$$

$$= \frac{1}{\sqrt{3}}$$

(xii) $\cos 225^\circ$

Solution:

We know that

$$2k\pi + \theta = \theta$$

$$\cos\left(225^\circ\right) = \cos\left(180^\circ + 45^\circ\right)$$

$$= \cos 45^\circ$$

$$= -\frac{1}{\sqrt{2}}$$

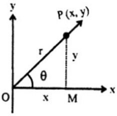
$$\because \left(\text{In quad,II } \cos\theta < 0\right)$$

Trigonometric Identities:

Consider an angle ∠MOP = θ radian in standard position. Let point P (x, y) be on the terminal side of the angle. By Pythagorean theorem, we have from right triangle OMP.

$$\mathrm{OM}^2+\mathrm{MP}^2=\mathrm{OP}^2$$

$$x^2+y^2=r^2 \quad .....(i)$$



Dividing both sides by r^2 we get

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$$

$$\Rightarrow\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1$$

$$\Rightarrow(\cos\theta)^2+(\sin\theta)^2=1$$

$$\therefore \boxed{\cos^2\theta+\sin^2\theta=1} \qquad (1)$$

Dividing (i) by x^2, we have

$$\frac{x^2}{x^2}+\frac{y^2}{x^2}=\frac{r^2}{x^2}$$

$$\Rightarrow \quad 1+\left(\frac{y}{x}\right)^2+\left(\frac{r}{x}\right)^2 \qquad \because \tan\theta=\frac{y}{x} \text{ and } \sec\theta=\frac{r}{x}$$

$$\Rightarrow \quad 1+(\tan\theta)^2+(\sec\theta)^2=1$$

$$\therefore \quad \boxed{1+\tan^2\theta=\sec^2\theta} \text{ or } \sec^2\theta-\tan^2\theta=1 \qquad (2)$$

Again dividing both sides of (i) by y^2, we get

$$\frac{x^2}{y^2}+\frac{y^2}{y^2}=\frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2+1=\left(\frac{r}{y}\right)^2 \qquad \because \cot\theta=\frac{x}{y} \text{ and } \mathrm{cosec}\theta\frac{r}{y}$$

$$\Rightarrow \quad (\cot\theta)^2+1=(\mathrm{cosec}\theta)^2$$

$$\therefore \quad \boxed{1+\cot^2\theta=\mathrm{cosec}^2\theta} \quad \text{or} \quad \mathrm{cosec}^2\theta-\cot^2\theta=1 \quad (3)$$

The identities (1), (2) and (3) are also known as Pythagorean Identities.

The fundamental identities are used to simplify expressions involving trigonometric functions.

Example 1:

Verify that cotθ secθ = cosecθ

Solution:

Expressing left hand side in terms of sine and cosine, we have

$$L.H.S \;=\cot\theta\sec\theta=\frac{\cos\theta}{\sin\theta}.\frac{1}{\cos\theta}$$

$$=\frac{1}{\sin\theta}=\mathrm{cosec}\theta$$

$$=\text{R.H.S}$$

Example 2

Verify that $\tan^4\theta+\tan^2\theta=\tan^2\theta\sec^2\theta$

Solution:

$$\text{L.H.S} = \tan^4\theta+\tan^2\theta=\tan^2\theta\,(\tan^2\theta+1) \qquad \because \tan^2\theta+1=\sec^2\theta$$

$$=\tan^2\theta\sec^2\theta$$

$$=\text{R.H.S}$$

Example 3

Show that $\frac{\cot^2\alpha}{\mathrm{cosec}\,\alpha-1}=\mathrm{cosec}\,\alpha+1$

Solution:

$$\frac{\cot^2\alpha}{\mathrm{cosec}\,\alpha-1} \qquad \begin{pmatrix}\because \mathrm{cosec}^2\theta-\cot\theta=1\\ \cot^2\theta=\mathrm{cosec}^2\theta-1\end{pmatrix}$$

$$=\frac{(\mathrm{cosec}^2\alpha-1)}{\mathrm{cosec}\,\alpha-1}=\frac{(\mathrm{cosec}\,\alpha-1)(\mathrm{cosec}\,\alpha+1)}{(\mathrm{cosec}\,\alpha-1)}=\mathrm{cosec}\,\alpha+1=R.H.S$$

Example 4

Express the trigonometric functions in terms of tan θ.

Solution:

By using reciprocal identity, we can express cotθ in terms of tanθ.

i.e., $\cot\theta=\frac{1}{\tan\theta}$

By solving the identity $1+\tan^2\theta=\sec^2\theta$

We have expressed secθ in terms of tanθ.

$$\sec\theta=\pm\sqrt{\tan^2\theta+1}\because$$

$$\cos\theta=\frac{1}{\sec\theta}\Rightarrow\cos\theta=\frac{1}{\pm\sqrt{\tan^2\theta+1}}$$

Because $\sin\theta=\tan\theta\cos\theta$, we have

$$\sin\theta=\tan\theta\left(\frac{1}{\pm\sqrt{\tan^2\theta+1}}\right)=\frac{\tan\theta}{\pm\sqrt{\tan^2\theta+1}}$$

$$\mathrm{cosec}\,\theta=\frac{1}{\sin\theta}=\frac{\pm\sqrt{\tan^2\theta+1}}{\tan\theta}$$

Exercise 7.4

In Problems 1- 6, simplify each expression to a single trigonometric function.

1. $\frac{\sin^2 x}{\cos^2 x}$

Solution:

$$\frac{\sin^2 x}{\cos^2 x}=\left(\frac{\sin x}{\cos x}\right)^2$$

$$=(\tan x)^2 \qquad \because \frac{\sin x}{\cos x}=\tan x$$

$$=\tan^2 x$$

2. tanx sinx secx

Solution:

$$\begin{aligned}\tan x\sin x\sec x &= \frac{\sin x}{\cos x}.\sin x.\frac{1}{\cos x}\\ &= \frac{\sin x.\sin x}{\cos x.\cos x}\\ &= \frac{\sin^2 x}{\cos^2 x}=\left(\frac{\sin x}{\cos x}\right)^2\\ &= (\tan x)^2 \qquad \because \frac{\sin x}{\cos x}=\tan x\\ &= \tan^2 x\end{aligned}$$

3. $\frac{\tan x}{\sec x}$

Solution:

$$\begin{aligned}\frac{\tan x}{\sec x} &= \frac{\sin x/\cos x}{1/\cos x}\\ &= \frac{\sin x}{\cos x}.\frac{\cos x}{1}\\ &= \sin x\end{aligned}$$

4. $1-\cos^2 x$

Solution:

$$\begin{aligned}1-\cos^2 x &= 1-(1-\sin^2 x) \qquad \because \cos^2 x = 1-\sin^2 x\\ &= 1-1+\sin^2 x\\ &= \sin^2 x\end{aligned}$$

5. $\sec^2 x-1$

Solution:

$$\begin{aligned}\sec^2 x-1 &= 1+\tan^2 x-1 \qquad \because \sec^2 x = 1+\tan^2 x\\ &= \tan^2 x\end{aligned}$$

6. $\sin^2 x.\cot^2 x$

Solution:

$$\begin{aligned}\sin^2 x.\cot^2 x &= \sin^2 x.\frac{\cos^2 x}{\sin^2 x} \qquad \because \cot x = \frac{\cos x}{\sin x}\\ &= \cos^2 x\end{aligned}$$

In problem 7 - 24, verify the identities.

7. $(1-\sin\theta)(1+\sin\theta)=\cos^2\theta$

Solution:

$$\begin{aligned}L.H.S &= (1-\sin\theta)(1+\sin\theta)\\ &= 1-\sin^2\theta\\ &= \cos^2\theta\\ &= R.H.S\end{aligned}$$

Hence proved.

8. $\dfrac{\sin\theta+\cos\theta}{\cos\theta}=1+\tan\theta$

Solution:

$$\begin{aligned} L.H.S &= \frac{\sin\theta+\cos\theta}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\cos\theta} \\ &= \tan\theta+1 \qquad \because \frac{\sin\theta}{\cos\theta}=1 \\ &= R.H.S \end{aligned}$$

Hence proved.

9. $(\tan\theta+\cot\theta)\tan\theta=\sec^2\theta$

Solution:

$$\begin{aligned} L.H.S &= (\tan\theta+\cot\theta)\tan\theta \\ &= \left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right).\frac{\sin\theta}{\cos\theta} \\ &= \left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right).\frac{\sin\theta}{\cos\theta} \\ &= \left(\frac{1}{\sin\theta\cos\theta}\right)\frac{\sin\theta}{\cos\theta} \qquad \because \sin^2\theta+\cos^2\theta=1 \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \\ &= R.H.S \end{aligned}$$

Hence proved.

10. $(\cot\theta+\operatorname{cosec}\theta)(\tan\theta-\sin\theta)=\sec\theta-\cos\theta$

Solution:

$$\begin{aligned} L.H.S &= (\cot\theta+\operatorname{cosec}\theta)(\tan\theta-\sin\theta) \\ &= \left(\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta}-\sin\theta\right) \\ &= \left(\frac{\cos\theta+1}{\sin\theta}\right)\left(\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}\right) \\ &= \left(\frac{1+\cos\theta}{\sin\theta}\right)\left(\frac{\sin\theta(1-\cos\theta)}{\cos\theta}\right) \\ &= \frac{(1+\cos\theta)(1-\cos\theta)}{\cos\theta} \\ &= \frac{1-\cos^2\theta}{\cos\theta} \end{aligned}$$

$$\begin{aligned}
&=\frac{1}{\cos\theta}-\frac{\cos^2\theta}{\cos\theta}\\
&=\frac{1}{\cos\theta}-\cos\theta\\
&=\sec\theta-\cos\theta\\
&=R.H.S
\end{aligned}$$

Hence proved.

11. $\frac{\sin\theta+\cos\theta}{\tan^2\theta-1}=\frac{\cos^2\theta}{\sin\theta-\cos\theta}$

Solution:

$$\begin{aligned}
L.H.S\ &=\frac{\sin\theta+\cos\theta}{\tan^2\theta-1}\\
&=\frac{\sin\theta+\cos\theta}{\frac{\sin^2\theta}{\cos^2\theta}-1}\\
&=\frac{\sin\theta+\cos\vartheta}{\frac{\sin^2\theta-\cos^2\theta}{\cos^2\theta}}\\
&=\frac{\cos^2\theta(\sin\theta+\cos\theta)}{\sin^2\theta-\cos^2\theta}\\
&=\frac{\cos^2\theta(\sin\theta+\cos\theta)}{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}\\
&=\frac{\cos^2\theta}{\sin\theta-\cos\theta}\\
&=R.H.S
\end{aligned}$$

Hence proved.

12. $\frac{\cos^2\theta}{\sin\theta}+\sin\theta=\operatorname{cosec}\theta$

Solution:

$$\begin{aligned}
L.H.S\ &=\frac{\cos^2\theta}{\sin\theta}+\sin\theta\\
&=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta}\\
&=\frac{1}{\sin\theta}\\
&=\operatorname{cosec}\theta\\
&=R.H.S
\end{aligned}$$

Hence proved.

13. $\sec\theta-\cos\theta=\tan\theta\sin\theta$

Solution:

$$\begin{aligned}
L.H.S\ &=\sec\theta-\cos\theta\\
&=\frac{1}{\cos\theta}-\cos\theta\\
&=\frac{1-\cos^2\theta}{\cos\theta}\\
&=\frac{\sin^2\theta}{\cos\theta}\\
&=\frac{\sin\theta}{\cos\theta}.\sin\theta\\
&=\tan\theta\sin\theta\\
&=R.H.S
\end{aligned}$$

Hence proved.

14. $\dfrac{\sin^2\theta}{\cos\theta}+\cos\theta=\sec\theta$

Solution:

$$\begin{aligned}L.H.S &= \frac{\sin^2\theta}{\cos\theta}+\cos\theta\\ &= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta}\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\cos\theta}\\ &= \sec\theta\\ &= R.H.S\end{aligned}$$

Hence proved.

15. $\tan\theta+\cot\theta=\sec\theta\operatorname{cosec}\theta$

Solution:

$$\begin{aligned}L.H.S &= \tan\theta+\cot\theta\\ &= \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\\ &= \frac{1}{\sin\theta\cos\theta}\\ &= \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\\ &= \operatorname{cosec}\theta\sec\theta\\ &= R.H.S\end{aligned}$$

Hence proved.

16. $(\tan\theta+\cot\theta)(\cos\theta+\sin\theta)=\sec\theta+\operatorname{cosec}\theta$

Solution:

$$\begin{aligned}L.H.S &= (\tan\theta+\cot\theta)(\cos\theta+\sin\theta)\\ &= \left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)(\cos\theta+\sin\theta)\\ &= \left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)(\cos\theta+\sin\theta)\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{\sin\theta\cos\theta}\right)(\cos\theta+\sin\theta)\\
&= \frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}\\
&= \frac{\cos\theta}{\sin\theta\cos\theta}+\frac{\sin\theta}{\sin\theta\cos\theta}\\
&= \frac{1}{\sin\theta}+\frac{1}{\cos\theta}\\
&= \operatorname{cosec}\theta+\sec\theta\\
&= R.H.S
\end{aligned}
$$

Hence proved.

17. $\sin\theta(\tan\theta+\cot\theta)=\sec\theta$

Solution:

$$
\begin{aligned}
L.H.S &= \sin\theta(\tan\theta+\cot\theta)\\
&= \sin\theta\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)\\
&= \sin\theta\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)\\
&= \sin\theta\left(\frac{1}{\cos\theta\sin\theta}\right)\\
&= \frac{1}{\cos\theta}\\
&= \sec\theta\\
&= R.H.S
\end{aligned}
$$

Hence proved.

18. $\dfrac{1+\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}=2\operatorname{cosec}\theta$

Solution:

$$
\begin{aligned}
L.H.S &= \frac{1+\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}\\
&= \frac{(1+\cos\theta)^2+(\sin\theta)^2}{\sin\theta(1+\cos\theta)}\\
&= \frac{1+2\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta(1+\cos\theta)}\\
&= \frac{1+2\cos\theta+1}{\sin\theta(1+\cos\theta)}\\
&= \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}\\
&= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\\
&= \frac{2}{\sin\theta}\\
&= 2\operatorname{cosec}\theta\\
&= R.H.S
\end{aligned}
$$

Hence proved.

19. $\dfrac{1}{1-\cos\theta}+\dfrac{1}{1+\cos\theta}=2\operatorname{cosec}^2\theta$

Solution:

$$\begin{aligned} L.H.S &= \frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta} \\ &= \frac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)} \\ &= \frac{2}{1-\cos^2\theta} \\ &= \frac{2}{\sin^2\theta} \\ &= 2\text{cosec}^2\theta \\ &= R.H.S \end{aligned}$$

Hence proved.

20. $\frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta}=4\tan\theta\sec\theta$

Solution:

$$\begin{aligned} L.H.S &= \frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta} \\ &= \frac{(1+\sin\theta)^2-(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} \\ &= \frac{(1+2\sin\theta+\sin^2\theta)-(1-2\sin\theta+\sin^2\theta)}{1-\sin^2\theta} \\ &= \frac{1+2\sin\theta+\sin^2\theta-1+2\sin\theta-\sin^2\theta}{\cos^2\theta} \\ &= \frac{4\sin\theta}{\cos^2\theta} \\ &= 4\frac{\sin\theta}{\cos\theta}.\frac{1}{\cos\theta} \\ &= 4\tan\theta\sec\theta \\ &= R.H.S \end{aligned}$$

Hence proved

21. $\sin^3\theta=\sin\theta-\sin\theta\cos^2\theta$

Solution:

$$\begin{aligned} R.H.S &= \sin\theta-\sin\theta\cos^2\theta \\ &= \sin\theta(1-\cos^2\theta) \\ &= \sin\theta.\sin^2\theta \\ &= \sin^3\theta \\ &= L.H.S \end{aligned}$$

Hence proved.

22. $\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)$

Solution:

$$
\begin{aligned}
L.H.S &= \cos^4\theta-\sin^4\theta\\
&=\left(\cos^2\theta\right)^2-\left(\sin^2\theta\right)^2\\
&=\left(\cos^2\theta-\sin^2\theta\right)\left(\cos^2\theta+\sin^2\theta\right)\\
&=\left(\cos^2\theta-\sin^2\theta\right)(1)\\
&=\cos^2\theta-\sin^2\theta\\
&=R.H.S
\end{aligned}
$$

Hence proved.

23. $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}=\dfrac{\sin\theta}{1-\cos\theta}$

Solution:

$$
\begin{aligned}
L.H.S &= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\\
&=\sqrt{\frac{1+\cos\theta}{1-\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}}\\
&=\sqrt{\frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)^2}}\\
&=\sqrt{\frac{1-\cos^2\theta}{(1-\cos\theta)^2}}\\
&=\sqrt{\frac{\sin^2\theta}{(1-\cos\theta)^2}}\\
&=\frac{\sqrt{\sin^2\theta}}{\sqrt{(1-\cos\theta)^2}}\\
&=\frac{\sin\theta}{1-\cos\theta}\\
&=R.H.S
\end{aligned}
$$

Hence proved.

24. $\sqrt{\dfrac{\sec\theta+1}{\sec\theta-1}}=\dfrac{\sec\theta+1}{\tan\theta}$

Solution:

$$
\begin{aligned}
L.H.S &= \sqrt{\frac{\sec\theta+1}{\sec\theta-1}}\\
&=\sqrt{\frac{\sec\theta+1}{\sec\theta-1}\times\frac{\sec\theta+1}{\sec\theta+1}}\\
&=\sqrt{\frac{(\sec\theta+1)^2}{(\sec\theta-1)(\sec\theta+1)}}\\
&=\sqrt{\frac{(\sec\theta+1)^2}{\sec^2\theta-1}}\\
&=\frac{\sqrt{(\sec\theta+1)^2}}{\sqrt{\tan^2\theta}}\\
&=\frac{\sec\theta+1}{\tan\theta}\\
&=R.H.S
\end{aligned}
$$

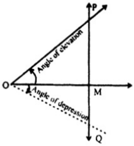
Hence proved.

Angle of Elevation and Angle of Depression:

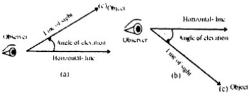
One of the objects of trigonometry is to find the distances between points or the heights of objects, without actually measuring these distances or heights.

Angle of elevation: Suppose O, P and Q are three points, P being at a higher level of O and Q being at lower level than O. Let a horizontal line drawn through O meet in M, the vertical line drawn through P and Q.

The angle MOP is called the angle of elevation of point P as seen from O. For looking at Q below the horizontal line we have to lower our eyes and ∠MOQ is called the angle of depression.



We measure an angle of elevation from a horizontal line up to an object or an angle of depression from a horizontal line down to an object,

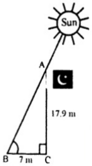


Find angle of elevation and angle of depression.

For finding distances height and angles by the use of trigonometric functions, consider the following examples:

Example 1

A flagpole 17.9-meter-high casts a 7-meter shadow. Find the angle of elevation of the sun.



Solution:

From the figure, we observe that is the angle of elevation.

Using the fact that

$$\tan\alpha = \frac{AC}{BC} = \frac{17.9}{7} \approx 2.55714$$

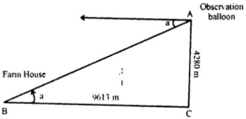
Solving for α gives us

$$\begin{aligned} \alpha &= \tan^{-1}(2.55714) \\ &= (68.6666)^\circ \approx 68^\circ 40' \\ \Rightarrow \quad \alpha &\approx 68^\circ 40' \end{aligned}$$

Example 2

An observation balloon is 4280 meter above the ground and 9613 meters away from a farmhouse. Find the angle of depressor of the farmhouse as observed from the observation balloon.

Solution:



For problems of this type the angle of elevation of A from B is considered equal to the

angle of depression of B from A, as shown in the diagram.

$$\begin{aligned} \tan\alpha &= \frac{AC}{BC} = \frac{4280}{9613} = 0.44523 \\ \alpha &= \tan^{-1}(0.44523) = 24^\circ \end{aligned}$$

So, angle of depression is 24°.

Exercise 7.5

1. Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow.

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that

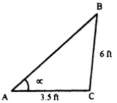
$$\tan \propto = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan \propto = \frac{6}{3.5}$$

$$\tan \propto = 1.7143$$

$$\propto = \tan^{-1}(1.7143)$$

$$\propto = 59.74^\circ$$



2. A tree casts a 40-meter shadow when the angle of elevation of the sun is 25°. Find the height of the tree.

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that

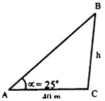
$$\tan \propto = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan 25^\circ = \frac{h}{40}$$

$$h = 40 \times \tan 25^\circ$$

$$= 40(0.44663)$$

$$= 18.652m$$



3. A 20 feet long ladder is leaning against a wall. The bottom of the ladder is 5 feet from the base of the wall. Find the acute angle (angle of elevation) the ladder makes with the ground.

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that

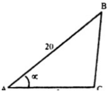
$$\cos \propto = \frac{\overline{AC}}{\overline{AB}}$$

$$\cos \propto = \frac{5}{20}$$

$$\propto = \cos^{-1}(0.25)$$

$$\propto = 75.5^\circ$$

$$or \propto = 75^\circ 30'$$



4. The base of a rectangle is 25 feet and the night of the rectangle is 13 feet. Find the angle that the diagonal of the rectangle makes with the base.

Solution:

From the figure, we observe that ∝ is the angle of elevation.

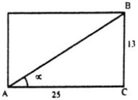
Using the fact that

$$\tan \propto = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan \propto = \frac{13}{25}$$

$$\propto = \tan^{-1}(0.52)$$

$$\propto = 27.47^\circ$$



5. A rocket is launched and climbs at a constant angle of 80°. Find the altitude of the rocket after it travels 5000 meters.

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that

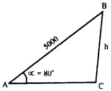
$$\sin \propto = \frac{\overline{BC}}{\overline{AB}}$$

$$\sin 80^\circ = \frac{h}{5000}$$

$$h = 5000 \times \sin 80^\circ$$

$$h = 5000(0.9848)$$

$$h = 4924.04\,m$$



6. An aero plane pilot flying at an altitude of 4000m wishes to make an approach to an airport at an angle of 50⁰ with the horizontal. How far from the airport will the plane be when the pilot begins to descend?

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that

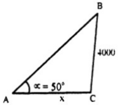
$$\tan \propto = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan 50^\circ = \frac{4000}{x}$$

$$x = \frac{4000}{\tan 50^\circ}$$

$$x = \frac{4000}{1.1918}$$

$$x = 3356.4\,m$$

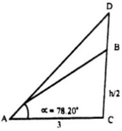


7. A guy wire (supporting wire) runs from the middle of a utility pole to the ground. The wire makes an angle of 78.2° with the ground and touch the ground 3 meters from the base of the pole. Find the height of the pole.

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that



$$\tan \propto = \frac{\overline{\text{BC}}}{\overline{\text{AC}}}$$

$$\tan 78.2^\circ = \frac{h/2}{3}$$

$$\frac{h}{2} = 3 \times \tan 78.2^\circ$$

$$\frac{h}{2} = 3(4.7867)$$

$$h = 6(407867)$$

$$h = 28.72\,m$$

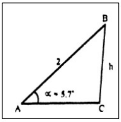
8. A road is inclined at an angle 5.7°. Suppose that we drive 2 miles up this road starting from sea level. How high above sea level are we?

Solution:

From the figure, we observe that ∝ is the angle of elevation.

Using the fact that

$$\sin \propto = \frac{\overline{\text{BC}}}{\overline{\text{AB}}}$$



$$\sin 5.7^\circ = \frac{h}{2}$$

$$h = 2\cos 5.7^\circ$$

$$= 2(0.099)$$

$$= 0.198\,\text{miles}$$

9. A television antenna of 8 feet height is located on the top of a house. From a point on the ground the angle of elevation to the top of the house is 17° and the angle of elevation to the top of the antenna is 21.8°. Find the height of the house.

Solution:

In □ABC

$$\tan 17^\circ = \frac{\overline{\text{BC}}}{\overline{\text{AC}}}$$

$$0.3057 = \frac{h}{\text{AC}}$$

$$\text{AC} = \frac{h}{0.3057} \qquad (1)$$

In □ADC

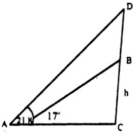
$$\tan 21.8^\circ = \frac{\overline{\text{CD}}}{\overline{\text{AC}}}$$

$$0.4 = \frac{\overline{\text{BC}} + \overline{\text{BD}}}{\overline{\text{AC}}}$$

$$0.4 = \frac{h+8}{\text{AC}}$$

$$\text{AC} = \frac{h+8}{0.4} \qquad (2)$$

Comparing equation (1) and (2), we have



$$\frac{h}{0.3057} = \frac{h+8}{0.4}$$

$$0.4h = 0.3057(h+8)$$

$$0.4h = 0.3057h + 2.4456$$

$$0.4h - 0.3057h = 2.4456$$

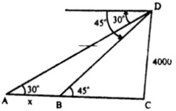
$$0.0943h = 2.4456$$

$$h = \frac{2.4456}{0.0943}$$

$h = 25.94$ feet

10. From an observation point, the angles of depression of two boats in line with this: point are found to 30° and 45°. Find the distance between the two boats if the point of observation is 4000 feet high.

Solution:



In □BCD

$$\tan 45^\circ = \frac{\overline{CD}}{\overline{BC}}$$

$$1 = \frac{4000}{\overline{BC}}$$

$$BC = 4000$$

In □ACD

$$\tan 30^\circ = \frac{\overline{CD}}{\overline{AC}}$$

$$\frac{1}{\sqrt{3}} = \frac{4000}{\overline{AB} + \overline{BC}}$$

$$\frac{1}{\sqrt{3}} = \frac{4000}{x + 4000}$$

$$x + 4000 = 4000\sqrt{3}$$

$$x = 4000\sqrt{3} - 4000$$

$$x = 6928.2 - 4000$$

$$x = 2928.2 \text{ feet}$$

11. Two ships, which are in line with the base of a vertical cliff, are 120 meters apart. The angles of depression from the top of the cliff to the ships are 30° and 45°, as show in the diagram.

(a) Calculate the distance $\overline{BC}$

(b) Calculate the height $\overline{CD}$, of the cliff.

Solution:

In □BCD

$$\tan 45^\circ = \frac{\overline{CD}}{\overline{BC}}$$

$$1 = \frac{\overline{CD}}{\overline{BC}}$$

$$\overline{BC}=\overline{CD} \qquad ....(1)$$

In □ACD

$$\tan 30^\circ=\frac{\overline{CD}}{\overline{AC}}$$

$$\frac{1}{\sqrt{3}}=\frac{\overline{CD}}{\overline{AB}+\overline{BC}}$$

$$\frac{1}{\sqrt{3}}=\frac{\overline{BC}}{\overline{AB}+\overline{BC}}$$

$$\frac{1}{\sqrt{3}}=\frac{\overline{BC}}{120+\overline{BC}} \qquad \because From(1)\overline{BC}=\overline{CD}$$

$$\sqrt{3}\,\overline{BC}=120+\overline{BC}$$

$$\sqrt{3}\,\overline{BC}-\overline{BC}=120$$

$$\left(\sqrt{3}-1\right)\overline{BC}=120$$

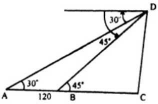
$$0.732\overline{BC}=120$$

$$\overline{BC}=\frac{120}{0.732}$$

$$\overline{BC}=164m$$

Put $\overline{BC}=164$ in eq(1), we get

$$\overline{CD}=164m$$



12. Suppose that we are standing on a bridge 30 feet above a river watching a log (piece of wood) floating toward we. If the angle with the horizontal to the front of the log is 16.7° and angle with the horizontal to the back of the log is 14°, how long is the log?

Solution:

In □BOC

$$\tan 16.7^\circ=\frac{\overline{OC}}{\overline{BC}}$$

$$0.3=\frac{30}{\overline{BC}}$$

$$\overline{BC}=\frac{30}{0.3}$$

$$\overline{BC}=100m$$

In □AOC

$$\tan 14^\circ=\frac{\overline{OC}}{\overline{AC}}$$

$$0.2493=\frac{30}{\overline{AB}+\overline{BC}}$$

$$0.2493=\frac{30}{x+100}$$

$$0.2493(x+100)=30$$

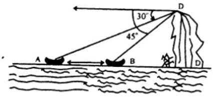
$$0.2493x+24.93=30$$

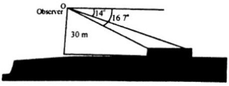
$$0.2493x=30-24.93$$

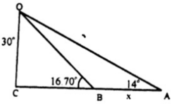
$$0.2493x=5.07$$

$$x=\frac{5.07}{0.2493}$$

$$x=20.33m$$







Miscellaneous Exercise – 7

Q1. Multiple Choice Question

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The union of two non-collinear rays, which have common end point is called

(a) an angle **(b)** a degree

(c) a minute **(d)** a radian

(ii) The system of measurement in which the angle is measured In radians is called

(a) CGS system **(b)** sexagesimal system

(c) MKS system **(d)** circular system

(iii) 20°

(a) 360' **(b)** 630'

(c) 1200' **(d)** 3600'

(iv) $\frac{3\pi}{4}$ **radians =**

(a) 115° **(b)** 135°

(c) 150° **(d)** 30°

(v) If $\tan\theta = \sqrt{3}$ **, then** θ **is equal to**

(a) 90° **(b)** 45°

(c) 60° **(d)** 30°

(vi) $\sec^2\theta =$

(a) $1 - \sin^2\theta$ **(b)** $I + \tan^2\theta$

(c) $1 + \cos^2\theta$ **(d)** $I - \tan^2\theta$

(vii) $\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$

(a) $2\sec^2\theta$ **(b)** $2\cos^2\theta$

(c) $\sec^2\theta$ **(d)** $\cos\theta$

(viii) $\frac{1}{2}$ **cosec45°**

(a) $\frac{1}{2\sqrt{2}}$ **(b)** $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ **(d)** $\frac{\sqrt{3}}{2}$

(ix) secθ cotθ =

(a) sinθ **(b)** $\frac{1}{\cos\theta}$

(c) $\frac{1}{\sin\theta}$ **(d)** $\frac{\sin\theta}{\cos\theta}$

(x) cosec²θ - cot²θ =

(a) -1 **(b)** 1

(c) 0 **(d)** tanθ

Answer:

(i)	a	**(ii)**	d	**(iii)**	c	**(iv)**	b	**(v)**	c
(vi)	b	**(vii)**	a	**(viii)**	b	**(ix)**	c	**(x)**	b

Q2. Write short answers of the following questions.

(i) Define an angle.

Ans:

An angle is defined as the union of two non-collinear rays with some common endpoint.

(ii) What is the sexagesimal system of measurement of angles?

Ans:

In sexagesimal system of measurement of angles, we find angles in degree, minutes and seconds.

(iii) How many minutes are in two right angles?

Ans:

In two right angles, we have 10800 minutes.

(iv) Define radian measure of an angle.

Ans:

The angle subtended at the center of the circle by an arc, whose length is equal to the radius of the circle, is called one radian.

(v) Convert $\frac{\pi}{4}$ radian to degree measure.

Ans:

$$\frac{\pi}{4}=\frac{\pi}{4}\text{radian}=\frac{\pi}{4}\times\frac{180^\circ}{\pi}=45^\circ$$

(vi) Convert 15° to radians.

Ans:

$$15^\circ=15\times1^\circ=15\times\frac{\pi}{180}=\frac{\pi}{12}\text{ radians.}$$

(vii) What is radian measure of the central angle of an arc 50m long on the circle of radius 25m.

Ans:

θ = **?**, l = 50m, r = 25m

We know that

$$l=r\theta$$
$$\theta=\frac{l}{r}=\frac{50}{25}=2\text{ radian}$$

(viii) Find r when l = 56 cm and 0 = 45°

Ans:

r = **?** , l = 56 cm,

$$\theta=45^\circ=45\times1^\circ=45\times\frac{\pi}{180}=\frac{\pi}{4}=0.7854\text{ radians}$$

We know that

$$l=r\theta$$
$$\mathrm{r}=\frac{l}{\theta}=\frac{56}{0.7854}$$
$$=71.30\,\text{cm}$$

(ix) Find tanθ when cosθ = $\frac{9}{41}$ and θ terminal side of the angle θ is in fourth quadrant.

Ans:

$$\cos\theta=\frac{9}{41}$$
$$\Rightarrow x=9\text{ and }\mathrm{r}=41$$

By Pythagoras theorem, we have

$$r^2=x^2+y^2$$
$$y^2=r^2-x^2$$
$$y=\sqrt{r^2-x^2}$$
$$=\sqrt{(41)^2-(9)^2}$$
$$=\sqrt{1681-81}$$
$$=\sqrt{1600}$$
$$=40$$

Now $\tan\theta=\frac{y}{x}=\frac{40}{9}$

(x) Prove that $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$:

$$\begin{aligned} \text{L.H.S} &= \left(1-\sin^2\theta\right)\left(1+\tan^2\theta\right) \\ &= \left(1-\sin^2\theta\right)\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right) \\ &= \left(\cos^2\theta\right)\left(\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right) \\ &= \left(\cos^2\theta\right)\left(\frac{1}{\cos^2\theta}\right) \end{aligned}$$

$$\begin{aligned} &= 1 \\ &= R.H.S \end{aligned}$$

Hence proved.

Q3. Fill in the blanks

(i) π radians =______degree.

(ii) The terminal side of angle 235° lies in __________quadrant.

(iii) Terminal side of the angle -30° lies in ___________quadrant.

(iv) Area of a circular sector is ___________.

(v) If r = 2 cm and θ = 3 radian, then area of the circular sector is _________.

(vi) The general form of the angle 480° is __________.

(vii) If $\sin\theta = \frac{1}{2}$, then θ = __________.

(viii) If θ = 300°, then sec (-300)° = _________.

(ix) $1 + \cot^2\theta$ = _________.

(x) Secθ — tanθ = ___________.

Answer

(i)	180	**(ii)**	III	**(iii)**	IV
(iv)	$\frac{1}{2}r^2\theta$	**(v)**	6cm ^{2}	**(vi)**	2kπ + 120°, where k = 1
(vii)	θ=30° or $\frac{\pi}{6}$rad	**(viii)**	2	**(ix)**	$\text{cosec}^2\theta$
(ix)	$\frac{1-\sin\theta}{\cos\theta}$				

Summary

✓ If we divide the circumference of a circle into 360 equal arcs, then the angle subtended at the center of the circle by one arc is called one degree and is denoted by 1°.

✓ The angle subtended at the center of the circle by an arc, whose length is equal to the radius of the circle, is called one radian.

✓ Relationship between radian and degree measure: $1^\circ = \frac{\pi}{180}\text{radian} \approx 0.0175$ radian and 1 radian = $\left(\frac{180}{\pi}\right)^\circ \approx 57.295$ degrees

✓ Relation between central angle and arc length of a circle: $l = r\theta$

✓ Area of a circular sector, $A = \frac{1}{2}r^2\theta$

✓ Two or more than two angles with the same initial and terminal sides are called coterminal angles.

✓ An angle is called a quadrantal angle, if its terminal side lies on the x-axis or y-axis.

✓ A general angle is said to be in standard position if its vertex is at the origin and its initial side is directed along the positive direction of the x-axis of a rectangular coordinate system.

✓There are six fundamental trigonometric ratios (functions) known as sine, cosine, tangent, cotangent, secant and cosecant.

✓ Trigonometric Identities:

(a) $\cos^2\theta + \sin^2\theta = 1$ **(b)** $1 + \tan^2\theta = \sec^2\theta$ **(c)** $1 + \cot^2\theta = \text{cosec}^2\theta$